Exploring Lagrangian and Hamiltonian Mechanics Through a Two-Degree-of-Freedom System: A Scaffolded Student Project Approach

Oksana Shevtsova[1]
[1]Department of Physical and Mathematical Sciences, National University "Kyiv-Mohyla Academy", Ukraine
*Corresponding Author: o.shevtsova@ukma.edu.ua

Received: 9th October 2025; **Accepted**: 18th November 2025; **Published**: 11th December 2025
DOI: https://dx.doi.org/10.29303/jpft.v11i2.10402

Abstract – *In this paper, we present a step-by-step description of a student project that illustrates the application of Lagrangian and Hamiltonian mechanics to a two-degree-of-freedom system. As part of the project, students explored both formalisms' peculiarities, identified conservation laws, and calculated integrals of motion. Students gained valuable experience by working with different approaches and observing the consistency with which they achieved equivalent results. The study has been fulfilled with a single cohort of fourteen undergraduate students. Each student received an individual version of the project with different initial conditions. Students uploaded the completed projects in a form of a brief report to a learning platform Moodle. The next stage was the teacher's review of the project and its evaluation. An evaluation was conducted for each of the three parts of the project, described in detail in the paper. After that, an individual meeting was held with each student to clarify the evaluation result. As part of this meeting, the student was surveyed about what was the most difficult and whether the result matched the expectations. Analysis of the assessment data showed a 20% improvement in conceptual understanding and demonstrated that all students could successfully apply both Lagrangian and Hamiltonian approaches to the physical system being modelled.*

Keywords: *Lagrangian and Hamiltonian formalisms; Conservation laws; Cyclic Coordinates; Poisson bracket*

INTRODUCTION

Lagrangian and Hamiltonian Formalisms are key structural elements in classical mechanics courses for undergraduate students and are a powerful part of the physics education culture. The Lagrangian and Hamiltonian approaches are completely equivalent, and it is easy to prove that each of them is indeed consistent with another. However each formalism is convenient and is applied behind the frame of classical mechanics (Goldstein H., Poole Ch. P., & Safko J., 2002), (Hamill P., 2014), (Marion J. B. & Thornton S.T., 2013), (Hand L.N., & Finch J.D., 1998), (Thorn C.B., 2013), (Jose J. V. & Saletan E. J., 1998)). Each technique has its own "playground" or physical space: configuration space (Lagrangian mechanics) and phase space (Hamiltonian mechanics) and its "key players": velocities and positions, and momenta and positions, respectively.

The aim of the project is to predict the dynamics of a planar particle applying the conservation laws, solving the Euler-Lagrange equations and Hamilton's equations, deriving the equations of motion, finding the integrals of motion for this system, visualizing the motion laws and the phase trajectory.

Learning Difficulties

Students often manipulate equations procedurally without conceptual understanding. They have difficulty transitioning between configuration and phase space descriptions. Students often struggle to identify conservation laws from the system symmetries and relate them to the integral of motion. In the project, students engaged in active problem-solving. However, its demanding nature required strong scaffolding (stepwise guidance, checkpoints, resources) written down in the paper and careful assessment strategies to ensure all students achieved meaningful outcomes rather than mechanically applying formulae.

Lagrangian

Thus, consider the Lagrangian of a planar particle (Landau L.D., Lifshitz E.M., & Rosenkevich L.V., 1935), (Landau L.D. & Lifschitz E.M., 1972)

$$L = \frac{\dot{x}^2 + \dot{y}^2}{x}, \tag{1}$$

where t is time (independent variable), $x = x(t)$ is a generalized coordinate, a function of time t, $y = y(t)$ is a generalized coordinate, a function of time t, $\dot{x} = \frac{dx}{dt}$ is the generalized velocity associated with x, $\dot{y} = \frac{dy}{dt}$ is the generalized velocity associated with y.

The Lagrangian (1) doesn't depend explicitly on y and t.

The initial conditions (ICs) are chosen as:

$$x(0) = 1, \dot{x}(0) = 1, y(0) = 0, \dot{y}(0) = 1\,. \tag{2}$$

A very important feature of the Lagrangian is that conserved quantities can easily be read off from it.

The generalized momentum "canonically conjugate" to the coordinate q_i is defined by a formula $p_i = \frac{\partial L}{\partial \dot{q}_i}$, i.e. the generalized momentum conjugate to a cyclic coordinate is a constant or a conserved quantity. This coordinate is known as "cyclic" or "ignorable". The Lagrangian (1) has cyclic coordinates t and y and it is easy to note them as coordinates that do not appear in the Lagrangian in explicit form.

The Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, i = 1,2 \tag{3}$$

for Lagrangian (1) can be written as:

$$\begin{cases} \dfrac{2\ddot{x}}{x} - \dfrac{\dot{x}^2}{x^2} + \dfrac{\dot{y}^2}{x^2} = 0 \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{y}} = \dfrac{dp_y}{dt} = \dfrac{d}{dt}\dfrac{2\dot{y}}{x} = 0 \Rightarrow \dfrac{2\dot{y}}{x} = C_1 \end{cases}, \tag{4}$$

with the ICs (2). The integration constant C_1 is determined from (4):

$$C_1 = \frac{2\dot{y}(0)}{x(0)} = 2\,. \tag{5}$$

Now, the first Euler-Lagrange equation (4) can be rewritten by considering ICs (2) and (5):

$$\frac{2\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} + 1 = 0\,. \tag{6}$$

Rewriting the Euler-Lagrange equation (6) gives:

$$\frac{2\ddot{x}}{x} - \frac{2\dot{x}^2}{x^2} = -\frac{\dot{x}^2}{x^2} - 1\,. \tag{7}$$

The equation (7) can be written as

$$\frac{d}{dt}\left(\frac{2\dot{x}}{x}\right) = -\frac{1}{4}\left(\frac{2\dot{x}}{x}\right)^2 - 1\,. \tag{8}$$

The generalized momentum p_x for Lagrangian (1) is equal to

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{2\dot{x}}{x}\,. \tag{9}$$

Thus, the differential equation (8) can be expressed as:

$$\frac{dp_x}{dt} + \frac{1}{4}p_x^2 = -1\,. \tag{10}$$

The differential equation (10) can be solved by separating the variables, yielding the solution:

$$p_x = \frac{2\cos t}{1 + \sin t}\,. \tag{11}$$

Using expression (11) and definition of the generalized momentum, the differential equation of motion can be presented as:

$$\frac{2\dot{x}}{x} = \frac{2\cos t}{1 + \sin t}\,. \tag{12}$$

Integration of the differential equation (12) with considering ICs (2) leads to the next laws of motion:

$$\begin{cases} x(t) = 1 + \sin t \\ y(t) = t - \cos t + 1 \end{cases}. \quad (13)$$

The $x(t)$ and $y(t)$ plots are shown in Figure 1. The trajectory $y(x)$ is shown in Figure 2. Trajectories $p_x(x)$ and $p_y(y)$ are shown in Figures 3a and 3b, respectively.

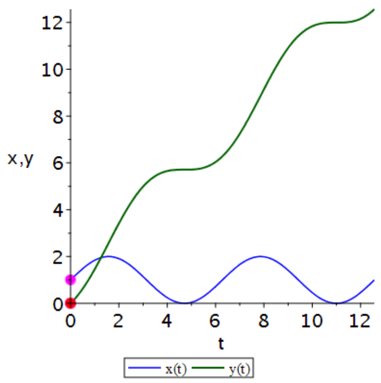

Figure 1.
The time dependence of $x(t)$ and $y(t)$. The red point indicates $y(0)$, the magenta point indicates the $x(0)$.

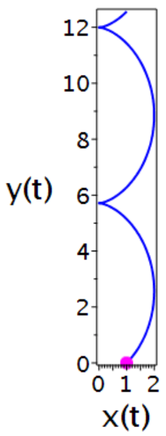

Figure 2.
The trajectory $y(x)$.The magenta point indicates the initial position $(x(0), y(0))$.

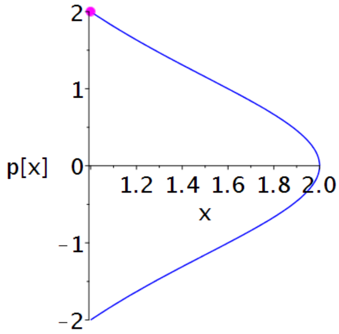

Figure 3a.
The phase trajectory $p_x(x)$.
The magenta point indicates the initial position $(x(0), p_x(0))$.

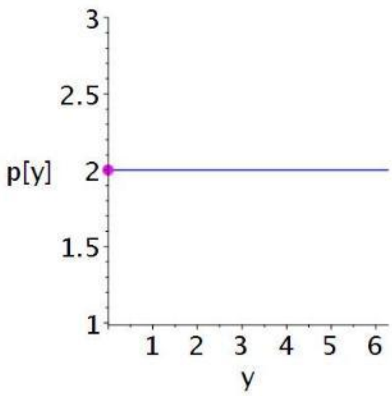

Figure 3b.
The phase trajectory $p_y(y)$.
The magenta point indicates the initial position $(y(0), p_y(0))$.

RESULTS AND DISCUSSION

The time dependence of $x(t)$ and $y(t)$ are presented in Fig 1. $x(t)$ depends sinusoidally (period $T = 2\pi$) on time. $y(t)$ stands for monotonic increase with small periodic fluctuations (linear and periodic component). The trajectory $y(x)$ in Fig. 2 and Fig. 4 describes cycloidal motion of the planar particle with detailed geometrical interpretation. Let's rewrite the laws of motion (13) in the time-parametric form:

$$\begin{cases} x(t) - 1 = \sin t \\ y(t) - t - 1 = -\cos t \end{cases}.$$

Using the Pythagorean identity, we obtain the standard form of a circle equation: $(x(t)-1)^2 + (y(t)-t-1)^2 = 1$ and interpret it geometrically. At each instant t the center of the circle is $(1, t+1)$. The radius is equal to 1. The point $(x(t), y(t))$ moves on a circle of radius 1, centered at $(1, t+1)$ (Fig.4). As t changes, the center $(1, t+1)$ moves upward the line $x = 1$ at a constant velocity, since y - coordinate increases linearly with t. Thus, the planar particle simultaneously moves around the moving center with angular position t and the motion is a cycloidal-type motion.

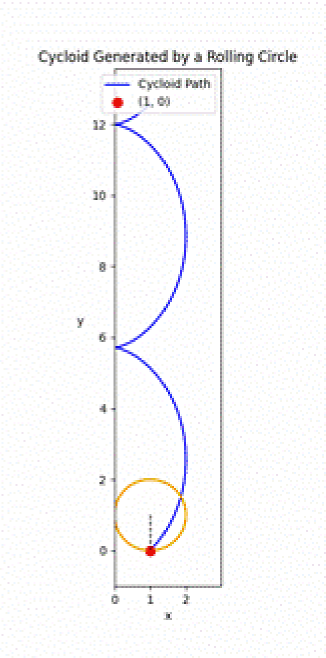

Figure 4.
Cycloid Generated by a Rolling Cycle from the initial position (1;0) .

The phase trajectory $p_x(x)$ in Fig.3a describes the dynamic of a planar particle in phase space. The phase trajectory $p_y(y)$ in Fig.3b illustrates that p_y does not depend on y, i.e. p_y is the integral of motion for the planar particle describing by Lagrangian (1).

Conservation Laws and Symmetries

Noether's theorem states that "for each symmetry of the Lagrangian, there exists a corresponding conserved quantity" (Cline D., 2017, Tatum J., 2025). For Lagrangian (1), the ignorable (cyclic) variables are t and y. Consequently, the conservation of momentum p_y arises when the Lagrangian is independent of y (Fig.3b). In other words, this conservation law results from the spatial translation invariance in the y direction. Thus,

$$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{2\dot{y}}{x} = const \qquad (14)$$

is a conserved quantity.

Conservation of energy arises when the Lagrangian is independent of time, i.e. $\frac{\partial L}{\partial t} = 0$. The law of conservation of energy can then be expressed as:

$$E = \sum_{i=1}^{2} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L = \frac{\dot{x}^2 + \dot{y}^2}{x} = const\,. \qquad (15)$$

Therefore, two integrals of motion, E and p_y, are obtained from the initial conditions (2):

$$p_y = 2, E = \frac{(\dot{x}(0))^2 + (\dot{y}(0))^2}{x(0)} = 2\,. \qquad (16)$$

Using the law of conservation of energy (16), the laws of motion can be obtained without directly solving the equations of motion. The procedure consists in solving the first-order differential equation with separated variables for $x(t)$:

$$\dot{x} = \frac{dx}{dt} = \pm\sqrt{2x - x^2}\,. \qquad (17)$$

Considering the direction of motion, i.e. knowing the value of x component of the initial velocity ($\dot{x}(0) = 1$), one can write the first-order differential equation with separated variables:

$$dt = \frac{dx}{\sqrt{2x - x^2}}\,. \qquad (18)$$

The integration of equation (18) leads to $t(x)$ dependency:

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{2x - x^2}} = \arcsin(x-1)\,, \qquad (19)$$

which can be rewritten as:

$$x(t) = 1 + \sin t\,. \qquad (20)$$

Then, knowing $x(t)$, the first-order differential equations (4) can be solved to find $y(t)$:

$$\frac{\dot{y}}{x}=1 \Rightarrow y=t-\cos t+1. \quad (21)$$

Thus, applying the energy conservation law leads to the same results (see the previous section). A visualization of these results is presented in Fig.1, Fig.2, Fig.3a and Fig.3b.

Hamiltonian Formalism

The "playground" in this case is defined as the six-dimensional phase space of position and momentum components.

Starting with Lagrangian (1), one can calculate the momentum components:

$$p_x=\frac{\partial L}{\partial \dot{x}}=\frac{2\dot{x}}{x},\, p_y=\frac{\partial L}{\partial \dot{y}}=\frac{2\dot{y}}{x}, \quad (22)$$

then invert these expressions to find the functions $\dot{x}(x,y,p_x,p_y)$, $\dot{y}(x,y,p_x,p_y)$, and now write the Hamiltonian for this planar particle $H(x,y,p_x,p_y)$ by using Legendre transformation:

$$H=\dot{x}p_x+\dot{y}p_y-L=\frac{1}{4}p_x^2x+\frac{1}{4}p_y^2x. \quad (23)$$

Then we rewrite the energy expression in the same variables:

$$E=\frac{1}{4}p_x^2x+\frac{1}{4}p_y^2x=H. \quad (24)$$

The energy coincides with Hamiltonian (23) and it is to prove that energy is an integral of motion, using the Poisson bracket.

The Poison Bracket as A Symmetry Identifier

In Hamiltonian mechanics, the Poisson bracket is an important binary operation, playing a central role in Hamilton's equations of motion, which govern the time evolution of a Hamiltonian dynamical system. The Poisson bracket is a very elegant and powerful tool in Hamiltonian mechanics that acts as a tool for symmetry analysis. Using the definition of Poisson bracket, anti-symmetry, linearity, the Leibniz rule, and the Jacobi identity, it is easy to find the integrals of motion in the phase space.

These constants of motion will commute with the Hamiltonian under the Poisson bracket. Suppose some function *f*(*p*, *q*) is a constant of motion. This implies that if $p(t)$, $q(t)$ is a trajectory or solution to Hamilton's equations of motion, then along that trajectory:

$$\frac{df}{dt}=0. \quad (25)$$

In particular, it is easy to prove that:

$$\{E,H\}=0,\{p_y,H\}=0=0. \quad (26)$$

Thus, E and p_y are integrals of motion.

Hamilton Canonical Equations of Motion

Hamilton's canonical equations of motion describe the time evolution of the canonical variables (q,p) in the phase space and can be written as:

$$\begin{cases} \dot{p}_x=-\dfrac{\partial H}{\partial x}=-\dfrac{1}{4}p_x^2-1 \\ \dot{x}=\dfrac{\partial H}{\partial p_x}=\dfrac{1}{2}p_x x \\ \dot{p}_y=-\dfrac{\partial H}{\partial y}=0 \Rightarrow p_y=2 \\ \dot{y}=\dfrac{\partial H}{\partial p_y}=\dfrac{1}{2}xp_y=x \end{cases} \quad (27)$$

Solutions of system (27) can be written in the form of (11), (13) and (16). So, the obtained results indicate that generalized momentum p_y and energy E are integrals of motions, and obviously, their values coincide with previous results.

CONCLUSION

Main outcomes of this project are conceptual and analytical skill gains of the sophomores of the Faculty of Natural Sciences of the National University "Kyiv-

Mohyla Academy" that participated in this project.

Main goals of the project were development of deep conceptual understanding using Lagrangian vs. Hamiltonian formulations of mechanics at describing the same physical system. Students involved in the project strengthened understanding terminology and improved their problem-solving & critical thinking, computational and visualization tools, and scientific communication skills.

Our limitations of the project were small sample size, that may restrict the generalizability of findings. A larger sample would provide greater statistical power and more general conclusions about population.

Future research should compare the effectiveness of this approach with traditional lecture-based instruction and computational-first methods.

CONFLICT OF INTEREST

The author declares that she has no conflict of interest and confirms full compliance with all ethical standards for scholarly research.

REFERENCES

Goldstein H., Poole Ch. P., & Safko J., (2002). *Classical Mechanics, 3rd Edition, Pearson Education, p. 665,* From https://physicsgg.me/wp-content/uploads/2014/12/classical_mechanics_goldstein_3ed.pdf.

Hamill P., (2014). *A Student's Guide to Lagrangians and Hamiltonians, Cambridge University Press, p. 186.* From https://scispace.com/pdf/a-student-s-guide-to-lagrangians-and-hamiltonians-fomv78omii.pdf.

Marion J. B. & Thornton S.T., (2013), *Classical dynamics of particles and systems. Fifth edition. p. 670, Brooks/Cole-Thomson Learning,* From https://eacpe.org/content/uploads/2016/11/Classical-Dynamics-of-Particles-and-Systems.pdf.

Hand L.N., & Finch J.D., (1998). *Analytical Mechanics, Cambridge University Press, p. 590*, From https://cdn.preterhuman.net/texts/science_and_technology/physics/Mechanics/Analytical%20mechanics%20-%20Hand,%20Finch.pdf

Thorn C.B., (2013). *Intermediate Classical Mechanics,* p. 56. From https://www.phys.ufl.edu/~thorn/homepage/cmuglectures.pdf

Jose J. V. & Saletan E. J., (1998). *Classical Dynamics – a Modern Approach*, *Cambridge Univ. Press,* From https://www.academia.edu/60585803/Jorge_V_Jos%C3%A9_Eugene_J_Saletan_Classical_Dynamics_A_Contemporary_Approach_Cambridge_University_Press_1998_.

Landau L.D. & Lifschitz E.M., (1972), *Mechanics, p. 284, Pergamon Press*. From https://pierre.ag.gerard.web.ulb.be/textbooks/books/Landau_Lifshitz_T1_short.pdf

Landau L.D., Lifshitz E.M., & Rosenkevich L.V., (1935). *Tasks on Theoretical Physics*, JSTI, State Scientific and Technical Edition of Ukraine. Kharkov, 1935. From https://escriptorium.karazin.ua/items/e70c0028-765a-4ace-9332-e88175d21f8d.

Cline D., (2017). *Variational Principles in Classical Mechanics, 1st Edition, University of Rochester River CAMPUS Libraries, 2017, p. 717.* From https://freecomputerbooks.com/Variational-Principles-in-Classical-Mechanics.html#google_vignette.