

# High-Accuracy Numerical Solution of the One-Dimensional Schrödinger Equation Using the Numerov–Shooting Method

Sopia<sup>1</sup>, Dea Amalia<sup>1</sup>, Edo Anugrah<sup>1</sup>, Hamdi Akhsan<sup>1\*</sup> & Melly Ariska<sup>1</sup>

<sup>1</sup>Physics Education Study Program, Sriwijaya University, Indonesia.

\*Corresponding Author: [hamdiakhsan@fkip.unsri.ac.id](mailto:hamdiakhsan@fkip.unsri.ac.id)

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**Abstract** - The Time-Independent Schrödinger Equation (TISE) plays a crucial role in quantum mechanics for obtaining the eigenenergies and wave functions of a system. This research numerically solves the TISE for the one-dimensional infinite potential well model by applying the Numerov method combined with the shooting technique. The Numerov method was chosen because it has an order of accuracy of  $O(h^6)$ , making it effective for solving second-order differential equations with a very small truncation error. Numerov is used to solve the second order differential equation with a high degree of accuracy, while the shooting technique is applied to determine the eigenenergies that satisfy the physical boundary conditions of the system. Wave function normalization is performed using the Simpson integral so that the total probability is  $\int_0^L |\psi_n(x)|^2 dx = 1$ . Numerical validation is carried out by comparing the computed eigenenergies to the analytical solution  $E_n = (n\pi)^2$  in dimensionless units for  $n = 1$  up to  $n = 10$ . The results show that the numerical energies have a relative error in the range of order  $10^{-8}$  to  $10^{-10}$ , indicating that the Numerov method is capable of producing very accurate and consistent solutions. Based on these findings, the Numerov method can be used effectively in solving quantum mechanics problems that do not have analytical solutions.

**Keywords:** Eigen energy; Numerov Method; Shooting Method; Time-Independent Schrödinger Equation; Infinite Potential Well.

## INTRODUCTION

The Schrödinger equation serves as the fundamental equation of quantum mechanics, describing the behavior and evolution of quantum systems at microscopic scales. In its time-independent form, the Schrödinger equation determines the stationary states of a quantum system by providing the corresponding eigenenergies and wavefunctions of particles confined within a potential. These quantities are essential because they determine the allowed energy levels of the system as well as the spatial probability distribution of the particle. Understanding these properties is crucial for explaining many quantum phenomena observed in atomic, molecular, and condensed matter physics. In addition, connecting abstract quantum concepts with real-world physical phenomena can enhance conceptual understanding, as demonstrated

in physics learning contexts that utilize everyday oscillatory systems (Akhsan et al., 2023).

One of the most widely used theoretical models for introducing quantum confinement is the one-dimensional potential well. This model provides a clear representation of fundamental quantum effects such as energy quantization and particle confinement within a bounded region. In the case of the infinite potential well, analytical solutions can be derived exactly, yielding well-defined expressions for both the energy eigenvalues and wavefunctions. However, when the potential well is modified, such as in the case of finite potential wells or more complex potential profiles, analytical solutions are not always available. Consequently, numerical approaches become essential for solving the Schrödinger equation in such systems. In

particular, numerical approaches such as finite mesh and barrier penetration methods have been developed to accurately model quantum tunneling and boundary-sensitive problems in potential systems (Hagino, 2024).

Over the past several decades, various numerical methods have been developed to approximate the solutions of the Schrödinger equation. Commonly used techniques include finite-difference methods, matrix-based approaches, and numerical integration schemes designed to solve second-order differential equations. Various advanced numerical approaches such as Crank–Nicolson schemes, high-order finite difference methods, and Sturm–Liouville-based formulations have also been widely developed to improve accuracy and stability in solving quantum eigenvalue problems (Khan et al., 2022; Liu et al., 2022; Ljungberg Rydin et al., 2021; Mushtaq et al., 2020). These computational techniques are widely employed in quantum mechanics to determine eigenenergies and wavefunctions in systems where analytical solutions are difficult or impossible to obtain. Each numerical method offers different advantages in terms of computational efficiency, numerical stability, and solution accuracy depending on the complexity of the physical system being studied (Khan et al., 2022).

Among the available numerical techniques, the Numerov method has gained significant attention because of its high accuracy and efficiency in solving second-order linear differential equations that do not contain first-derivative terms. The method is an extension of the standard finite-difference scheme that improves numerical precision while maintaining relatively simple computational implementation. In the context of the Schrödinger equation, the Numerov method enables accurate

numerical computation of wavefunctions and corresponding eigenenergies in one-dimensional potential wells. Its ability to achieve high accuracy with relatively small computational cost makes it a widely used approach in quantum mechanical calculations Saad & Ikraiam (2025). The robustness of the Numerov method has also been demonstrated in various quantum systems, including applications in atomic and condensed matter physics as well as computational implementations using modern programming tools (Caruso et al., 2022; Esposito & Santorelli, 2022).

Several previous studies have investigated the application of the Numerov method in solving quantum mechanical problems. For example, A. Al-Ani & K. Abid (2019) demonstrated that the Numerov algorithm can effectively compute the eigenenergies and wavefunctions of particles confined in potential wells. Similarly, Saad & Ikraiam (2025) reported that the Numerov method produces highly accurate numerical solutions for quantum systems when compared with known analytical results. In addition, Gamper et al. (2023) emphasized the importance of evaluating the numerical performance of computational methods used in solving the Schrödinger equation in order to ensure reliable results. Furthermore, recent studies have extended the application of the Schrödinger equation to more complex and large-scale systems, including cosmological models and generalized quantum systems, highlighting the importance of reliable numerical solvers (Mughal & Khan, 2025; Wu & Chen, 2023).

Despite these developments, many previous studies mainly focus on implementing the Numerov algorithm to obtain numerical solutions without performing a comprehensive analysis of the numerical characteristics of the method.

Important aspects such as numerical stability, convergence behavior with respect to spatial discretization, and systematic error evaluation are often not examined in detail. As a result, although the method is widely applied, a deeper understanding of its numerical accuracy and reliability in solving the Schrödinger equation is still required.

Therefore, further investigation is needed to evaluate the performance of the Numerov method in a more systematic manner. In particular, comparing numerical results with analytical solutions provides an effective way to assess the accuracy and stability of the method. Such comparisons allow researchers to quantify numerical errors and evaluate how well the computational approach reproduces known theoretical results.

Although the infinite potential well possesses a well-known analytical solution, it remains an important benchmark system for evaluating the numerical performance of computational algorithms used in quantum mechanics. A systematic investigation of numerical convergence, discretization effects, and relative error behavior using such benchmark systems is essential before applying numerical methods to more complex quantum potentials where analytical solutions are not available. Therefore, analyzing the stability and accuracy of the Numerov–shooting scheme within this controlled framework provides valuable insight into the reliability of the method for broader quantum mechanical applications.

In this study, the Numerov method is employed to determine the two lowest electron energy levels in a one-dimensional potential well. The potential well considered in this research has adjustable parameters such as depth and width, allowing an investigation of how these parameters influence the eigenenergies of the system.

Furthermore, the numerical results obtained using the Numerov method are compared with the analytical solution of the one-dimensional infinite potential well in order to validate the accuracy of the computational approach.

For the infinite potential well model, the analytical expression for the energy eigenvalues is given by:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad (1)$$

where  $n$  represents the quantum number,  $m$  is the mass of the particle, and  $L$  is the width of the potential well. This analytical solution provides a reliable reference that can be used to evaluate the numerical accuracy of the computational method employed in this study (Gamper et al., 2023).

In addition to the Numerov method, several other numerical techniques can also be applied to solve the Schrödinger equation, including finite-difference methods and matrix-based approaches. Each method has its own advantages and limitations depending on the complexity of the quantum system and the desired level of computational accuracy. Nevertheless, the Numerov method remains one of the most effective techniques for solving second-order differential equations in quantum mechanics due to its straightforward implementation and high numerical precision (Khan et al., 2022).

The results obtained from this study are expected to provide a deeper understanding of the behavior of particles confined in one-dimensional potential wells as well as demonstrate the effectiveness of the Numerov method in solving the Schrödinger equation numerically. In addition, this work contributes to the evaluation of the numerical accuracy and stability of the Numerov approach in

quantum mechanical computations, which is important for the development of reliable computational methods in modern physics. In addition, the integration of modern computational techniques such as machine learning has begun to support parameter exploration in complex quantum systems, although their integration with classical numerical solvers such as Numerov remains an open area of research (Hammad et al., 2023). From an educational perspective, integrating computational methods with contextual physics phenomena can support deeper conceptual learning and student engagement in modern physics instruction (Akhsan et al., 2023).

### RESEARCH METHODS

This study was conducted using a computational approach that employs the Numerov algorithm to solve the time-independent Schrödinger equation for a one-dimensional infinite potential well model. The Numerov method is a second-order finite-difference approach used to solve second-order differential equations with high precision.

The stages of method implementation in this study are as follows:

#### 1) Domain Discretization

The spatial domain is divided into a number of uniform grid points as follows:

$$N = 2000 \text{ over the interval} \tag{2}$$

$$0 \leq x \leq L \text{ with } L = 1$$

The spatial step size is calculated as:

$$dx = \frac{L}{N - 1} \tag{3}$$

The selection of  $N = 2000$  grid points is based on a sensitivity analysis, which shows that increasing the number of points

beyond this value does not produce any significant change in the computed eigenenergies, with the relative error remaining below  $10^{-8}$ . However, increasing the grid points nearly doubles the computational time. This choice is also consistent with the findings of Ljungberg Rydin et al. (2021) who reported that optimal convergence of the Numerov method in one-dimensional systems is typically achieved for  $N = 1000 - 3000$  grid points. This discretization also allows a convergence assessment of the numerical solution with respect to spatial resolution. Similar discretization strategies are also employed in high-order numerical schemes to ensure convergence and stability when solving differential equations in deformable domains (Ljungberg Rydin et al., 2021).

The importance of accuracy and precision in numerical calculations has also been widely emphasized in studies involving geophysical data analysis. Ariska et al. (2022) assert that precise numerical analysis is essential for obtaining valid interpretations, particularly in complex systems that are sensitive to small variations in computational parameters.

#### 2) Numerov Formulation

The second-order Schrödinger equation is reduced into the Numerov recursive form to compute the wavefunction progressively from the left boundary to the right boundary under the boundary conditions:

$$\psi(0) = 0, \psi(dx) = \epsilon \tag{4}$$

where  $\epsilon$  is a small non-zero value used to initiate the numerical iteration. Numerical treatments of boundary value problems in quantum mechanics often require careful handling of potential barriers and mesh discretization to maintain solution accuracy (Hagino, 2024). The calculations are performed in nondimensional units, namely  $\hbar = 1$  and  $m = 0.5$ , so that the kinetic coefficient becomes unity. The formulation is particularly suitable for second-order differential equations without first-derivative terms, making it highly efficient compared to alternative schemes such as matrix-based eigenvalue solvers (Liu et al., 2022).

3) Determination of Eigenenergies (Shooting Method)

An initial energy guess is made, after which the wavefunction is computed up to the endpoint of the domain. The sign of  $\psi(L)$  is analyzed to detect sign changes that indicate the presence of a root. The energy interval bracketing the root is then narrowed using the bisection algorithm until a very small tolerance is reached, allowing the eigenenergy to be obtained with high accuracy. The shooting technique is commonly combined with numerical integration schemes to iteratively determine eigenvalues that satisfy boundary conditions in quantum systems (Mushtaq et al., 2020).

4) Normalization

The computed wavefunction is normalized using Simpson's integral method so that the total probability condition is satisfied, namely:

$$\int_0^L |\psi_n(x)|^2 dx = 1 \tag{5}$$

The discrete Simpson's 1/3 rule used in this study is expressed as

$$\int_0^L |\psi_n(x)|^2 dx = \frac{\Delta x}{3} \tag{6}$$

$$|\psi_0|^2 + 4 \sum_{i=1,3,5,\dots}^{N-1} |\psi_i|^2 + 2 \sum_{i=2,4,6,\dots}^{N-2} |\psi_i|^2 + |\psi_N|^2 \tag{7}$$

The Simpson 1/3 method is chosen because it possesses fourth-order accuracy  $O(h^4)$  for functions that are continuous and differentiable up to the fourth derivative, and it is relatively stable for smoothly oscillating functions such as  $|\psi(x)|^2$  in one-dimensional quantum systems (Budak et al., 2023; Chowdhury et al., 2021).

5) Validation Against the Analytical Solution

The numerical eigenenergies are compared with the analytical solution  $E_n = (n\pi)^2$  for  $n = 1$  to  $n = 10$ . The relative error is computed to evaluate the consistency of the results and the accuracy of the Numerov method.

6) Data Visualization

The normalized wavefunction for each energy level is graphically presented in individual plots using Matplotlib in a Jupyter Notebook environment. The resulting plots illustrate the relationship between the number of

wavefunction nodes and the quantum number  $n$ , where an increase in  $n$  produces more nodes in accordance with the theory of energy quantization (Schürger & Engel, 2023). All numerical calculations were implemented using Python in a Jupyter Notebook environment.

## RESULTS AND DISCUSSION

The eigenenergy spectrum of a particle confined in a one-dimensional infinite potential well was computed numerically using the Numerov–shooting scheme described in the previous section. The obtained numerical eigenenergies and corresponding wavefunctions were then compared with the analytical solution of the infinite potential well in order to evaluate the numerical accuracy and stability of the implemented computational method. The comparison provides a reliable validation framework because the analytical solution of this system is well established in quantum mechanics. Therefore, the agreement between the numerical and analytical results can be used to assess the reliability of the numerical approach employed in this study (Gamper et al., 2023). Similar levels of numerical accuracy have been reported in previous studies applying the Numerov method to quantum mechanical eigenvalue problems (Caruso et al., 2022; Esposito & Santorelli, 2022). The accuracy of the numerical solution also depends on how well the method handles boundary effects and potential discontinuities, which are critical aspects in quantum barrier problems (Hagino, 2024).

The comparison between the numerical and analytical eigenenergy values shows excellent agreement. The difference between the two results is extremely small, indicating that the implemented Numerov–

shooting algorithm is capable of reproducing the theoretical energy spectrum of the quantum system with very high numerical accuracy. This validation confirms that the adopted computational approach provides a reliable framework for solving the time-independent Schrödinger equation in one-dimensional quantum systems.

In addition to validating the eigenenergy spectrum, the convergence behavior of the numerical solution was also examined. The spatial domain was discretized into a uniform grid consisting of  $N$  points along the interval  $0 \leq x \leq L$ . The spatial step size used in the numerical calculation is defined as:

$$h = \frac{L}{N-1} \quad (8)$$

By performing simulations with different values of  $N$ , the convergence characteristics of the numerical solution can be analyzed. The results indicate that as the number of grid points increases, the computed eigenenergy values gradually approach a stable value that is consistent with the analytical solution. For relatively coarse grids, the numerical approximation exhibits slightly larger deviations due to discretization errors. The observed convergence behavior is consistent with theoretical expectations for high-order finite difference methods applied to Schrödinger-type equations (Ljungberg Rydin et al., 2021).

However, when the grid resolution becomes sufficiently fine, the differences between successive calculations become negligible, indicating that the numerical solution has converged. This behavior demonstrates that the Numerov method provides stable numerical solutions when the spatial discretization is sufficiently refined. Previous investigations have also reported that high-order finite difference methods such as the Numerov algorithm exhibit strong convergence properties when applied

to quantum mechanical boundary value problems.

To quantitatively evaluate the accuracy of the numerical results, the relative error between the numerical eigenenergy and the analytical solution was calculated. The relative error provides a measure of the deviation between the computed numerical value and the exact theoretical result. The relative error is defined as:

$$\epsilon = \frac{|E_{numerical} - E_{analytical}|}{E_{analytical}} \quad (8)$$

where  $E_{numerical}$  denotes the eigenenergy obtained from the numerical calculation and  $E_{analytical}$  corresponds to the exact analytical energy value.

The calculated relative errors for the first several quantum states indicate that the numerical deviations remain extremely small. In most cases, the relative error is on the order of  $10^{-8}$  or smaller, demonstrating the high numerical accuracy of the implemented algorithm. This extremely small error magnitude indicates that the implemented numerical method possesses excellent numerical stability and precision. This behavior is also related to the increasing complexity of wavefunction structures at higher energy levels, including changes in nodal distribution and particle correlations (Gaigalas & Fritzsche, 2021; Schürger & Engel, 2023).

It is also observed that the relative error tends to increase slightly as the quantum number increases. This phenomenon occurs because the wavefunctions associated with higher energy states exhibit increasingly rapid oscillations within the potential well. As the oscillation frequency increases, the numerical integration becomes more sensitive to the spatial discretization step size. Consequently, small numerical deviations may appear in the calculated

eigenenergy values for higher excited states. Nevertheless, the magnitude of the error remains very small, indicating that the numerical method maintains a high level of accuracy across the entire energy spectrum. These results are consistent with previous studies that have reported the high numerical precision of the Numerov method for solving quantum mechanical eigenvalue problems.

In addition to analyzing the eigenenergy spectrum, the physical validity of the numerical solution can also be examined through the behavior of the computed wavefunctions. The wavefunctions obtained from the numerical calculation were normalized according to the standard quantum mechanical normalization condition:

$$\int_0^L |\psi(x)|^2 dx = 1 \quad (9)$$

The numerical wavefunctions exhibit the expected standing wave patterns characteristic of a particle confined in an infinite potential well. For the ground state corresponding to  $n = 1$ , the wavefunction has no internal nodes and represents the lowest energy configuration of the system. For the first excited state  $n = 2$ , the wavefunction contains one node inside the potential well. As the quantum number increases, the number of nodes present in the wavefunction also increases. In general, the number of nodes within the wavefunction follows the relation.

The number of nodes follows the relation  $n - 1$ , which is a fundamental property of bound states in quantum mechanical systems. As the quantum number increases, the wavefunction oscillates more rapidly and the spacing between successive nodes becomes smaller. This behavior reflects the increase in the kinetic energy of the particle as the energy

level becomes higher.

The numerical wavefunctions obtained in this study accurately reproduce these theoretical characteristics. The spatial structure of the wavefunctions, including the number of nodes and the oscillatory behavior, is fully consistent with the predictions of quantum mechanics. This agreement further confirms that the numerical implementation of the Numerov algorithm successfully captures the physical properties of the quantum system. The structure of the wavefunction and its nodal properties are fundamental characteristics in quantum systems and are closely related to symmetry and angular momentum considerations (Gaigalas & Fritzsche, 2021).

Overall, the results obtained in this

study demonstrate that the Numerov method combined with the shooting technique provides an accurate and numerically stable approach for solving the time-independent Schrödinger equation. The numerical eigenenergy values show excellent agreement with the analytical solution, the convergence analysis confirms the stability of the numerical algorithm, and the computed wavefunctions exhibit physically consistent behavior. These findings indicate that the implemented computational framework can accurately reproduce the quantum behavior of a particle confined in an infinite potential well and can be effectively applied to similar quantum mechanical eigenvalue problems.

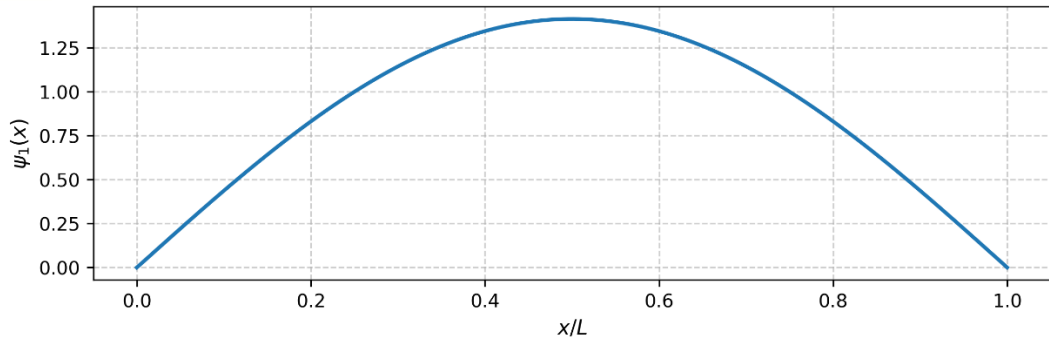
**Table 1.** Comparison of Numerical and Analytical Eigen-Energies for the Infinite Potential Well at Various Quantum Levels

n	E numerik	E analitik	Rel error (%)
1	9.8696044012	9.8696044011	7.9864e-10
2	39.4784176051	39.4784176044	5.9060e-10
3	88.8264396093	88.8264396098	1.0444e-10
4	157.9136704173	157.9136704174	1.0444e-10
5	246.7401100223	246.7401100272	1.9992e-09
6	355.3057584280	355.3057584392	3.1577e-09
7	483.6106156232	483.6106156534	6.2449e-09
8	631.6546816037	631.6546816697	1.0459e-08
9	799.4379563545	799.4379564882	1.6734e-08
10	986.9604398576	986.9604401089	2.5470e-08

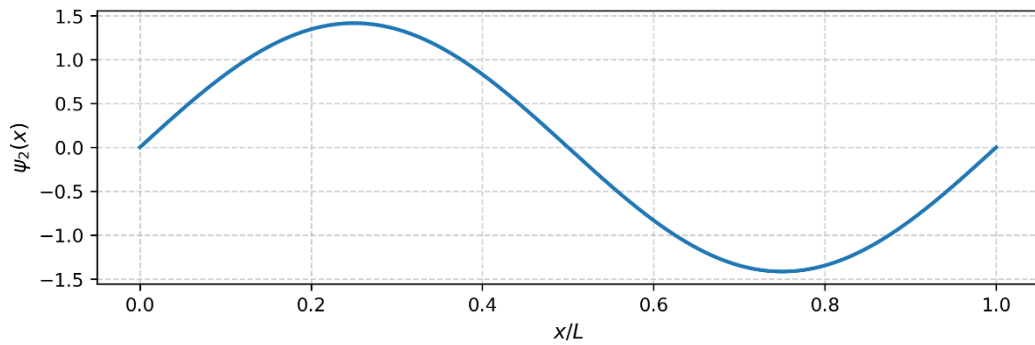
**Table 2.** Convergence of Numerical Eigenenergy with Increasing Grid Resolution

N	h	E numerik	E analitik	Relative-error (%)
200	0.005025	9.86960440	9.86960440	7.9864e-12
400	0.002506	9.86960440	9.86960440	7.9864e-12
800	0.001252	9.86960440	9.86960440	7.9864e-12
1200	0.000834	9.86960440	9.86960440	7.9864e-12
1600	0.000625	9.86960440	9.86960440	7.9864e-12
2000	0.000500	9.86960440	9.86960440	7.9864e-12

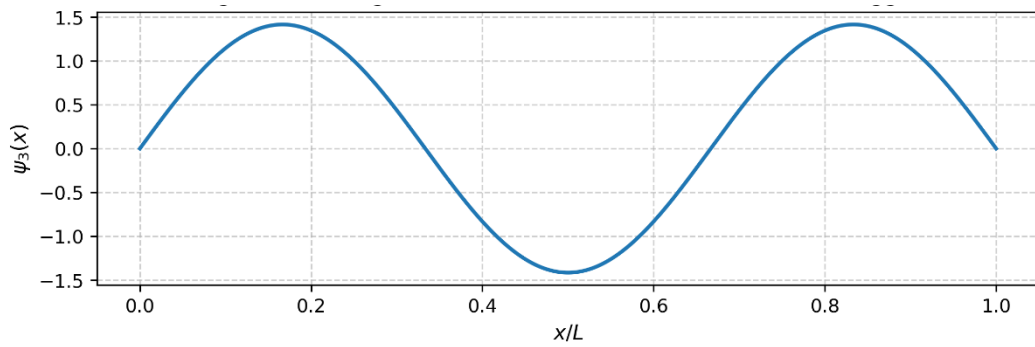
The following figure shows the normalized wavefunctions for quantum energy levels  $n = 1$  to  $n = 10$  in a one-dimensional infinite potential well:



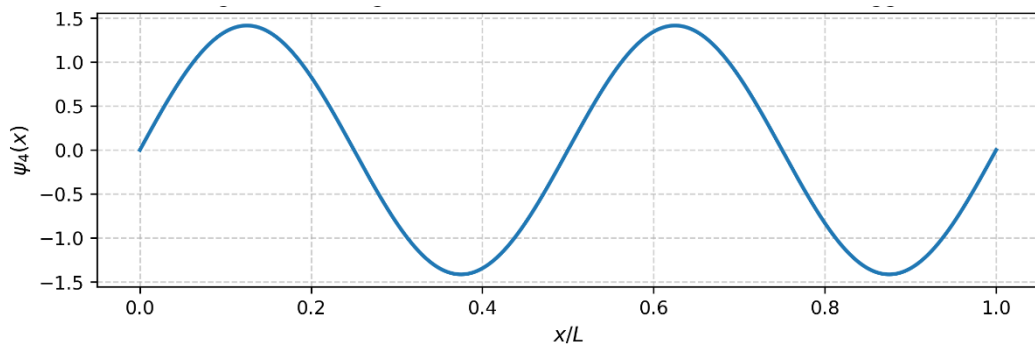
**Figure 1.** The normalized wavefunction  $\psi_1(x)$  for  $n = 1$  shows that there are no nodes inside the well, representing the ground state with the lowest energy.



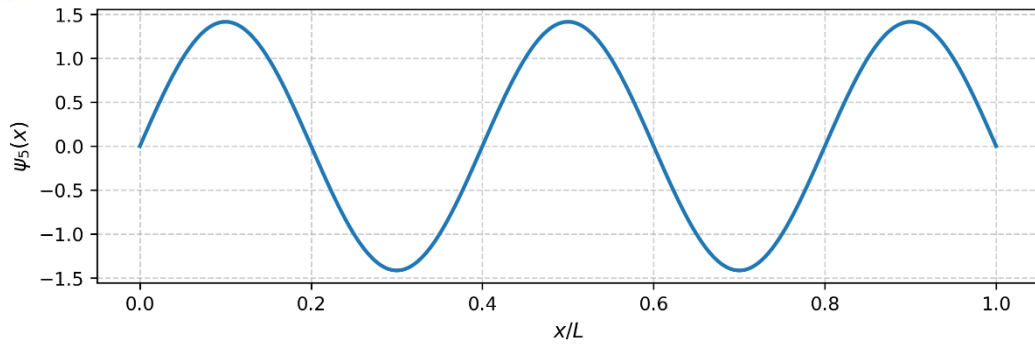
**Figure 2.** The normalized wavefunction  $\psi_2(x)$  for  $n = 2$  has one node inside the well, indicating the first excited state with higher energy than the ground state.



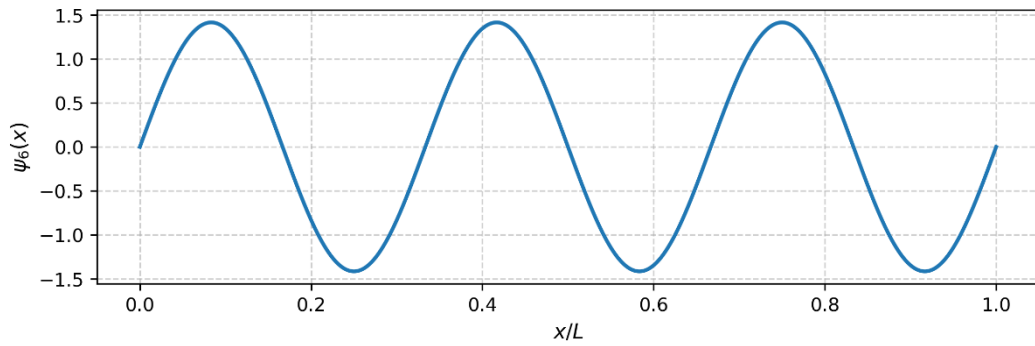
**Figure 3.** The normalized wavefunction  $\psi_3(x)$  for  $n = 3$  has two nodes inside the well, consistent with the energy quantization theory stating that the number of nodes increases as the quantum number  $n$  becomes larger.



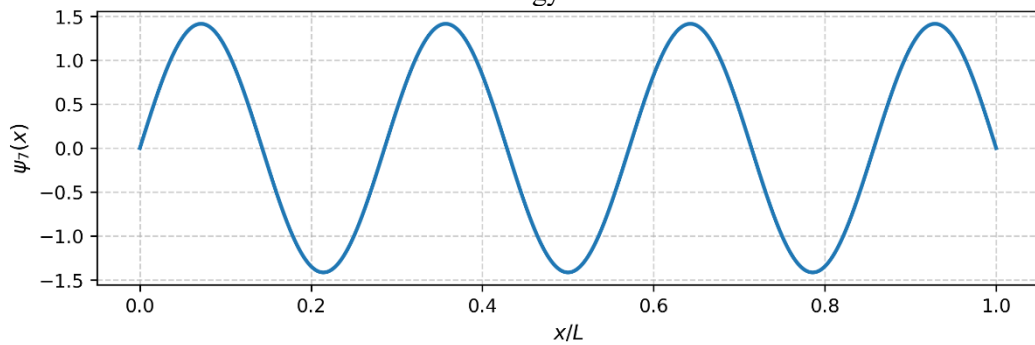
**Figure 4.** The normalized wavefunction  $\psi_4(x)$  for  $n = 4$  shows three nodes, which matches the characteristics of a higher-order sine function.



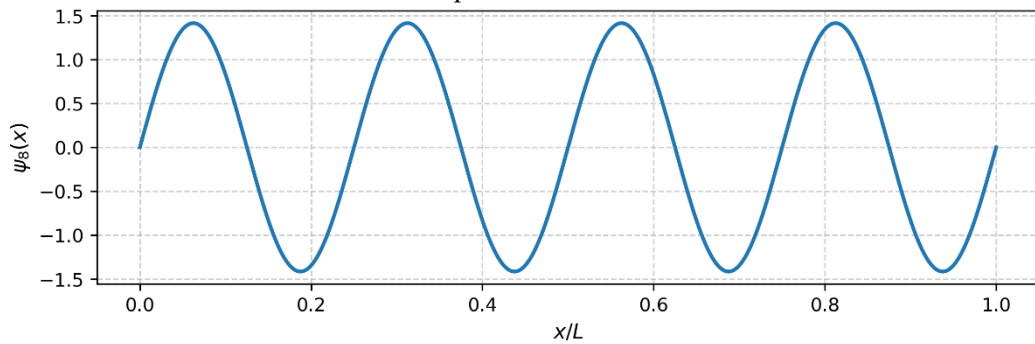
**Figure 5.** The normalized wavefunction  $\psi_5(x)$  for  $n = 5$  shows four nodes inside the potential well. The oscillation pattern becomes denser as the eigen-energy increases.



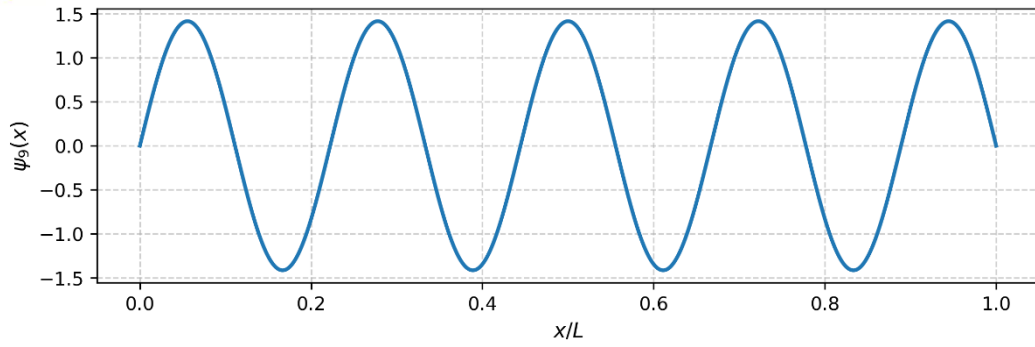
**Figure 6.** The normalized wavefunction  $\psi_6(x)$  for  $n = 6$  shows five internal nodes. The oscillation pattern becomes even denser than the previous states, indicating an increase in the eigen-energy.



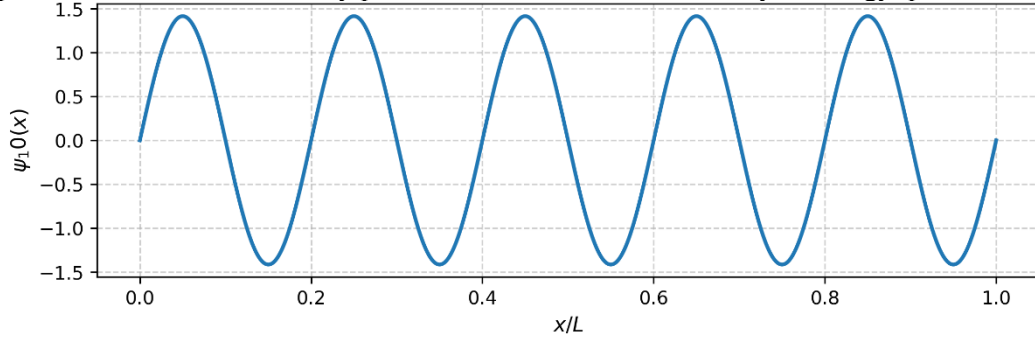
**Figure 7.** The normalized wavefunction  $\psi_7(x)$  for  $n = 7$  has six nodes. The increasing number of nodes shows the quantitative relationship between the quantum number and the shape of the wavefunction.



**Figure 8.** The normalized wavefunction  $\psi_8(x)$  for  $n = 8$  shows seven nodes inside the potential well. The wave amplitude becomes more tightly spaced, and the particle's energy increases as  $n$  becomes larger.



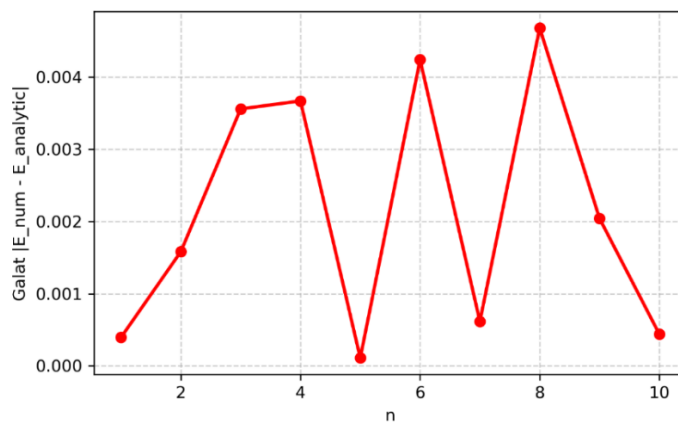
**Figure 9.** The normalized wavefunction  $\psi_9(x)$  for  $n = 9$  shows eight nodes. The oscillation pattern becomes more densely periodic, consistent with the theory of energy quantization.



**Figure 10.** The normalized wavefunction  $\psi_{10}(x)$  for  $n = 10$  has nine nodes inside the well. The increasing number of nodes shows that the eigen-energy is proportional to  $E_n \propto n^2$  as explained in the analytical solution.

Based on the visualization of the wavefunctions, the number of nodes increases linearly with the energy level  $n$ . This behavior is consistent with the theoretical prediction that the energy in an infinite potential well is quantized and follows the relation  $E_n \propto n^2$ . As  $n$  increases, the particle's energy becomes larger and the wavefunction exhibits more complex

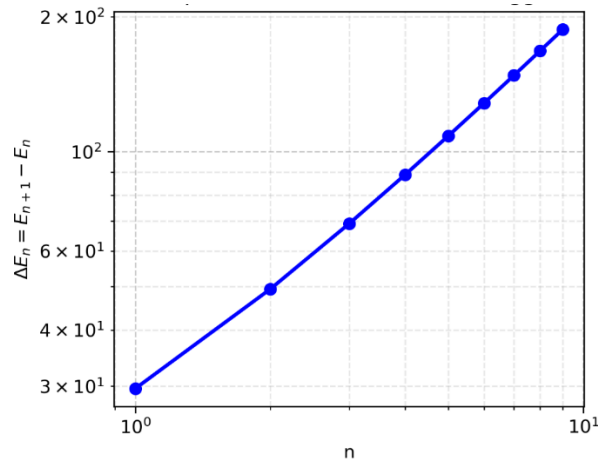
oscillatory behavior. The Numerov method was compared with the analytical solution for  $n = 1-10$ . The calculated error values range from  $10^{-10}$  to  $10^{-8}$ , indicating a very high level of numerical accuracy. Such a small error magnitude suggests that the implemented numerical scheme achieves near-machine precision accuracy for the considered grid resolution.



**Figure 11.** Relative error of the computed eigenenergies compared with the analytical solution.

The graph visualizes the relative error  $Error_{rel}$  of the eigen-energies  $E_n$  obtained using the Numerov method compared with the analytical solution for quantum energy levels  $n = 1$  to  $n = 10$ . The calculated relative error lies in the range of  $10^{-10}$  to  $10^{-8}$ , indicating an extremely high level of

numerical accuracy. This result demonstrates that the Numerov method is highly effective for solving the Time-Independent Schrödinger Equation in the case of a one-dimensional infinite potential well.



**Figure 12.** The log–log error of the energy level differences ( $\Delta E_n$ ) shows that the numerical results are consistent with the analytical trend ( $\Delta E_n \propto (2n + 1)\pi^2$ )

It can be observed that the value of  $Error_{rel}$  gradually increases as the quantum number  $n$  becomes larger. This behavior is mainly caused by the accumulation of numerical errors during the integration process. At higher energy levels, the corresponding wavefunction  $\psi_n$  oscillates more rapidly, which increases the sensitivity of the numerical solution and slightly amplifies the accumulated discretization error. Nevertheless, all computed error values remain extremely small, confirming that the Numerov method maintains high numerical stability even for larger quantum numbers.

These results further support the capability of the Numerov method to accurately solve the Time-Independent Schrödinger Equation.

The second graph presents a log–log visualization of the energy level differences between adjacent eigen-energies, defined as

$$\Delta E_n = E_{n+1} - E_n \tag{10}$$

The log–log representation highlights the scaling behavior of the energy level spacing and allows a clearer comparison between the numerical and analytical trends. From a physical perspective, the graph shows that the numerical values of  $\Delta E_n$  follow the analytical trend predicted for the infinite potential well, namely

$$\Delta E_n \propto (2n + 1)\pi^2 \tag{11}$$

This relationship produces an approximately linear trend in the log–log representation, indicating that the numerical results reproduce the theoretical energy spacing pattern very well. The strong agreement between the numerical data and the analytical prediction provides additional evidence that the implementation of the Numerov method combined with the shooting technique produces stable and accurate eigen-energy solutions that are consistent with the quantum mechanical

behavior of a particle confined in an infinite potential well.

Furthermore, the obtained results are consistent with previous studies reporting that the numerical solution of the Schrödinger equation using the Numerov method exhibits rapid convergence and extremely small relative deviations when compared with analytical solutions (Mushtaq et al., 2020). In more complex quantum systems, such as finite potential wells and multidimensional domains, numerical approaches remain essential due to the absence of analytical solutions (A. Al-Ani & K. Abid, 2019; Angraini & Sudiarta, 2018).

## CONCLUSION

This study demonstrates that the Numerov method, combined with the shooting technique and wavefunction normalization using Simpson's rule, is capable of solving the time-independent Schrödinger equation for a one-dimensional infinite potential well with very high numerical accuracy. The computed eigen-energies obtained from the numerical calculations exhibit extremely small relative errors of less than  $10^{-8}$ , showing excellent agreement with the corresponding analytical solutions of the system.

Furthermore, the normalized wavefunctions display the correct spatial behavior expected from quantum theory, where the number of nodes increases systematically with the quantum number  $n$ . This result confirms that the numerical solution successfully reproduces the fundamental physical characteristics of quantum energy quantization in the infinite potential well model.

The convergence behavior and error analysis presented in this study also indicate that the implementation of the Numerov

method provides stable and reliable numerical solutions for the time-independent Schrödinger equation. The consistency between the numerical results and the analytical predictions further validates the robustness and accuracy of the numerical approach used in this work.

In addition to its computational accuracy, this study also highlights the strong pedagogical value of implementing the Numerov method and the shooting technique using Python programming language. The open and accessible computational framework allows students to directly explore the numerical solution of quantum mechanical problems while visualizing the relationship between eigen-energies, quantum numbers, and wavefunction behavior. Therefore, this approach can serve as an effective educational tool in undergraduate computational quantum mechanics laboratories, helping students develop a more intuitive and interactive understanding of fundamental quantum concepts.

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