

# Influence of Gravity-Induced Axial Force on the Natural Frequencies of an Inclined Cantilever Beam

Ananta Kusuma Yoga Pratama\*, Sunarko

Department of Civil Engineering, Universitas 17 Agustus 1945 Banyuwangi, Banyuwangi, Indonesia

\*e-mail: [ananta@untag-banyuwangi.ac.id](mailto:ananta@untag-banyuwangi.ac.id)

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**Abstract:** The limitations of conventional horizontal cantilever models in capturing gravity-induced axial effects in inclined structures that influence stiffness and natural frequencies. This study investigates the influence of gravity on the dynamic behavior of an inclined cantilever beam, with particular emphasis on a 10-m steel beam configuration. Unlike conventional horizontal cantilever models, structural inclination generates a spatially varying axial compressive force that increases from zero at the free end to a maximum value at the fixed support. The objective of this study is to analyze how this gravity-induced axial force distribution affects the natural frequencies and effective stiffness of the beam. To achieve this, a high-fidelity numerical model was developed in MATLAB, where the beam was discretized into 100 Euler–Bernoulli elements to capture the gradual variation of axial load and the resulting geometric softening effect. The analytical formulation is based on the Rayleigh–Ritz method, in which the spatial axial force distribution  $N(x)$  is incorporated into the potential energy expression. The predicted natural frequencies were validated through frequency-domain analysis of the transient response obtained from numerical simulation. Both analytical and numerical results indicate that, for an inclination angle of  $60^\circ$ , the first and second natural frequencies are approximately 1.84 Hz and 11.61 Hz, respectively. These results demonstrate that beam inclination significantly reduces the effective structural stiffness due to gravity-induced geometric softening. The findings highlight the importance of incorporating spatially varying geometric stiffness in the dynamic analysis, stability assessment, and vibration-sensitive design of large-scale inclined cantilever structures.

**Keywords:** Geometric Stiffness; Inclined Cantilever Beam; Natural Frequency; Self-weight; Rayleigh–Ritz Method.

## Introduction

Accurate prediction of natural frequencies is essential in the dynamic analysis and design of slender structural elements such as beams, towers, and cantilever systems [1]–[3]. The vibrational characteristics of such structures are primarily governed by the distribution of mass, bending stiffness, and boundary constraints. In classical vibration theory, natural frequencies are obtained as eigenvalues of the governing equations of motion and therefore depend strongly on structural geometry, material properties, and the loading conditions acting on the system [4]–[6].

Among structural elements, the cantilever beam serves as a fundamental model for studying the dynamics of flexible structures [7]. Within the framework of continuum mechanics, its vibration behavior is typically described by partial differential equations incorporating inertia and bending stiffness effects. Classical analytical solutions provide the corresponding mode shapes and natural frequencies that characterize the system response [8], [9]. However, in many conventional beam vibration analyses, especially for horizontal cantilevers, self-weight is either neglected or modeled solely as a transverse distributed load along the beam [10], [11].

In practical engineering, many cantilevered components operate at an inclination, causing the gravitational force to deviate from the beam's principal axes [12], [13]. From a physical perspective, this inclination decomposes the gravitational vector into two distinct

components: a transverse load and an axial force. This axial component introduces a pre-stress effect that modifies the effective stiffness of the beam [14]. Consequently, gravity acts not merely as a static load but as a critical factor that alters the system's inherent dynamic properties, most notably its natural frequency spectrum [15].

Recent advances in beam have mainly addressed specialized beam configurations that differ from a simple inclined cantilever subjected to self-weight-induced axial loading. Nonlinear studies on vertical or hanging cantilever beams have investigated the influence of gravity on internal resonance, nonlinear coupling, and extreme-amplitude vibrations [12], [16]. Other investigations have focused on axially moving beams and related systems, where the dynamic stability and nonlinear response are governed by prescribed axial motion, moving loads, elastic foundations, or hygro-thermal effects [17], [18]. In parallel, axially loaded beams and frameworks have been analyzed using exact dynamic-stiffness or frequency-dependent stiffness and geometric-stiffness matrices, but the axial load is usually taken as constant (or a simple parameter) and not derived from a spatially varying gravity component along an inclined beam [19], [20] [19]–[21].

Despite the extensive literature on beam vibration and axially loaded structures, analytical and numerical studies that explicitly account for the spatially varying axial force generated by gravity in inclined cantilever beams remain limited [22][23]. In particular, the consistent decomposition of gravity into transverse and axial components along the

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beam length, and the incorporation of the resulting non-uniform geometric stiffness into the natural-frequency formulation, have not been sufficiently explored. Furthermore, the relationship between the theoretical formulation of such systems and high-resolution numerical discretization models has not been systematically investigated.

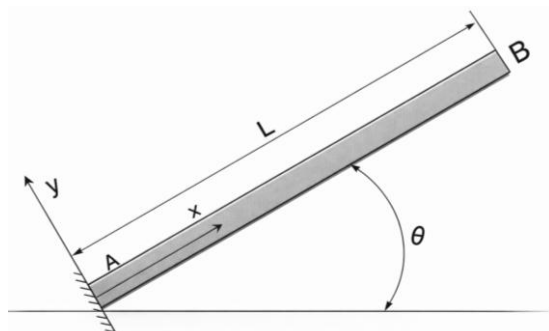
Therefore, the present study investigates the natural frequencies of an inclined cantilever beam subjected to self-weight by explicitly incorporating the gravity-induced axial force distribution into the governing formulation. A theoretical framework based on continuum mechanics is developed using the Rayleigh–Ritz method, in which the spatial distribution of axial force is included in the potential-energy formulation to account for geometric stiffness effects. The analytical formulation is complemented by numerical simulations using a refined Euler–Bernoulli beam discretization implemented in MATLAB. Through this combined theoretical and numerical approach, the influence of inclination angle and gravitational loading on the modal characteristics of the cantilever beam is systematically evaluated.

The results of this study provide new physical insights into the interaction between gravity-induced axial forces, geometric stiffness, and the dynamic behavior of inclined cantilever structures. The proposed framework also provides a consistent analytical basis for evaluating natural frequencies in structural systems where inclination and self-weight play significant roles, thereby supporting the dynamic design and stability assessment of engineering structures operating in inclined configurations.

## Research Methods

### Assumption and model parameter

The dynamic behavior of the inclined cantilever beam is formulated based on the classical Euler–Bernoulli beam theory [24]. The beam is assumed to be homogeneous, isotropic, and linearly elastic, and only small transverse deflections are considered. Under these assumptions, the effects of shear deformation and rotary inertia are neglected. The beam has a length  $L$ , density  $\rho$ , cross-sectional area  $A$ , and Young’s modulus  $E$ . The bending moment is therefore taken about the  $z$ -axis. For this reason, the flexural rigidity is expressed as  $EI_z$ , where  $I_z$  is the second moment of area about the bending axis. This notation is adopted to ensure consistency with the beam orientation and to avoid ambiguity in the definition of the bending stiffness.



**Figure 1.** Inclined cantilever beam with axial gravitational load.

The beam is inclined at an angle  $\theta$  with respect to the horizontal axis, shown in Figure 1. As a result, the gravitational force produces both transverse and axial components. The transverse component contributes to bending deformation, whereas the axial component generates a distributed axial force along the beam [25]. This axial force modifies the structure's effective stiffness, thereby influencing its natural frequencies.

### Analytical model

The natural frequency is determined using the Rayleigh method, which is widely applied in the vibration analysis of continuous systems [26]. The total potential energy consists of the elastic bending energy and the energy associated with the axial force induced by gravity.

The bending strain energy is expressed as:

$$U_b = \frac{1}{2} \int_0^L E I_z \left( \frac{\partial^2 v(x,t)}{\partial x^2} \right)^2 dx \tag{1}$$

where  $v(x, t)$  denotes the transverse displacement. The axial force produces an additional strain energy term:

$$U_a = \frac{1}{2} \int_0^L N(x) \left( \frac{\partial v(x,t)}{\partial x} \right)^2 dx \tag{2}$$

The kinetic energy of the beam is given by:

$$T = \frac{1}{2} \int_0^L \rho A \left( \frac{\partial v(x,t)}{\partial t} \right)^2 dx \tag{3}$$

Applying the condition  $T_{max} = U_{max}$ , the total potential energy of the system is therefore:

$$U = U_b + U_a \tag{4}$$

The axial force distribution is obtained from the equilibrium of an infinitesimal beam element, considering the component of the self-weight acting along the beam axis. The distributed weight per unit length is  $\rho Ag$ . The component along the beam axis is determined by the inclination angle. Integrating the axial component along the beam length and applying the boundary condition at the free end leads to [25]:

$$N(x) = -\rho Ag(L - x)\sin(\theta) \tag{5}$$

This expression indicates that the axial force varies linearly along the beam and reaches its maximum magnitude at the clamped end.

Assuming harmonic motion, the transverse displacement is expressed as  $v(x, t) = \phi(x)\cos(\omega t)$ , and substituting this expression into the energy formulation, the Rayleigh method yields the following frequency equation:

$$\omega^2 = \frac{\int_0^L E I_z \left( \frac{d^2 \phi}{dx^2} \right)^2 dx + \int_0^L N(x) \left( \frac{d\phi}{dx} \right)^2 dx}{\int_0^L \rho A \phi^2 dx} \tag{6}$$

This expression is used to approximate the natural frequency of the inclined cantilever beam. Alternatively, the governing equation of motion can be derived from the dynamic equilibrium of an infinitesimal beam element. This leads to:

$$EI_z \frac{d^4 \phi(x)}{dx^4} - N(x) \frac{d^2 \phi(x)}{dx^2} = \rho A \omega^2 \phi(x) \quad (7)$$

Substituting the axial force distribution results in:

$$EI_z \frac{d^4 \phi(x)}{dx^4} + \rho A g(L - x) \sin(\theta) \frac{d^2 \phi(x)}{dx^2} = \rho A \omega^2 \phi(x) \quad (8).$$

This formulation accounts for the influence of gravity and beam inclination on the natural frequencies.

**Numerical simulation.**

To verify the analytical prediction obtained from the Rayleigh–Ritz formulation, a numerical model of the inclined cantilever beam was developed. The governing equation of the system includes a spatially varying axial force term due to the gravitational component along the beam axis. This axial force originates from the beam's self-weight and accumulates along its length, increasing from the free end toward the fixed support. Consequently, the axial force becomes a function of the spatial coordinate, leading to a differential equation with variable coefficients. Such systems are generally difficult to solve analytically in closed form, particularly when the axial force varies along the beam length variable  $(L - x)$ . Therefore, a numerical discretization approach is adopted to enable the dynamic behavior of the beam to be evaluated computationally [27], [28].

In this approach, the continuous beam is represented as a discretized system composed of a finite number of Euler–Bernoulli beam elements [28]. The beam domain is divided into 100 elements along its length, allowing the spatial variation of the gravity-induced axial force to be represented with sufficient resolution while maintaining computational efficiency. Each element contributes to the global mass and stiffness matrices of the structure, resulting in a matrix-based representation of the beam dynamics.

The axial force distribution derived from the analytical formulation in Equation 5 is incorporated into the stiffness formulation to account for the effect of gravity-induced axial loading on the structural response. The clamped boundary condition at the fixed end of the cantilever beam is implemented by constraining both displacement and rotation degrees of freedom, while the free end remains unconstrained. This numerical discretization enables the time-domain vibration response of the inclined cantilever beam to be simulated and provides an independent computational framework for validating the natural frequencies predicted by the analytical Rayleigh–Ritz formulation.

**Simulation procedure**

The simulations were carried out for a range of inclination angles, including both negative and positive orientations with respect to the horizontal, in order to investigate the influence of the gravity-induced axial force on the dynamic behavior of the beam [29]. For each inclination angle, a small initial perturbation was introduced to excite the fundamental vibration mode. This approach ensures that the system undergoes free oscillations without external forcing, enabling the dominant natural frequency to be identified. The transverse displacement response of the

beam was recorded over time at the free end, where the dynamic response is most significant.

The time-domain data gathered from the numerical simulations were subsequently processed using the Fast Fourier Transform (FFT) algorithm in MATLAB. To ensure high spectral accuracy and prevent aliasing, the sampling frequency ( $F_s$ ) was carefully derived from the simulation's fixed time step, enabling precise mapping of the frequency content. The dominant peak in the power spectral density (PSD) was taken as the system's fundamental natural frequency. This frequency-domain approach provides high-fidelity estimates of the natural frequency and facilitates direct comparison with analytical predictions from the Euler–Bernoulli beam theory.

**Analytical Validation**

The numerical results were compared with the analytical solution obtained from the Rayleigh–Ritz method. In this study, the Rayleigh–Ritz formulation employs a polynomial shape function:

$$\phi(x) = \frac{x^2(3L-x)}{2L^3} \quad (9)$$

which satisfies the geometric boundary conditions of a cantilever beam — zero deflection and zero slope at the fixed end. The fundamental natural frequency is estimated by equating the maximum potential and kinetic energies of the system (Equation 6), where the numerator contains two contributions: the elastic strain energy associated with bending, and the potential energy modification due to the spatially varying axial force in Equation 5. The denominator represents the system's reference kinetic energy. The three integrals are evaluated numerically over the beam length for each inclination angle  $\theta$ . It is important to note that the Rayleigh–Ritz method, by virtue of using an approximate shape function, always provides an upper bound estimate of the true natural frequency.

The closer the assumed shape function is to the actual mode shape, the lower — and more accurate — the estimated frequency. This upper bound property makes the method a reliable tool for validating the FEM results: if both methods agree within an acceptable tolerance, it confirms that the FEM solution has converged to the true solution. This comparison allows the accuracy of the FEM formulation to be assessed independently of simulation parameters and provides physical insight into the influence of geometric stiffness on the variation in natural frequency with inclination angle.

**Results and Discussion**

In structural dynamics, conventional Rayleigh-Ritz formulations typically assume a uniform axial force along the beam's span. However, for inclined beams positioned at an angle  $\theta$ . This distribution becomes non-uniform due to self-weight. In the analyzed physical model, the compressive force at the free end (Point B) vanishes, whereas it reaches its maximum at the fixed support (Point A) as it bears the cumulative weight of the entire beam length.

This non-uniformity is mathematically represented in the potential energy equation, where the axial force  $N(x)$

remains within the integrand. The definition of  $N$  as a spatial function of  $(x)$  signifies that gravitational forces act differentially at every infinitesimal point along the beam span, shown in Figure 2. Analytically, solving this governing equation requires complex integration or the implementation of Bessel functions to obtain an exact solution. Consequently, the implementation of discretization serves as a highly realistic and essential methodology. By partitioning the beam into 100 discrete elements, the complex variation in axial force is approximated by assuming a constant force within each local segment, while allowing for incremental changes from the fixed support to the free tip. This approach ensures that the second-order effects of gravity are accurately captured within the numerical simulation.

```

29      54 13*Le 156 -22*Le; -13*Le -3*Le^2 -22*Le 4*Le^2);
30
31      for i = 1:n
32          x_mid = (i-0.5)*Le;
33          M_i = -rho*A*g*(L-x_mid)*sin(delta); % Eq.271: tekan, fungsi x
34          kg_i = (M_i/(30*Le))*[36 3*Le -36 3*Le; 3*Le 4*Le^2 -3*Le -Le^2; ...
35              -36 -3*Le 36 -3*Le; 3*Le -Le^2 -3*Le 4*Le^2];
36
37          idx = (2*i-1):(2*i+2);
38          K(idx,idx) = K(idx,idx) + ke_base;
39          Kg(idx,idx) = Kg(idx,idx) + kg_i;
40          M(idx,idx) = M(idx,idx) + me_base;
41      end
42
43      K_total = K + Kg;
44      K_free = K_total(3:end,3:end);
45      M_free = M(3:end,3:end);
46
47      %% 3. ANALISIS MODAL
48      [V, D] = eig(K_free, M_free);
49      wn = sqrt(diag(D));

```

Figure 2. The contribution of the axial force  $N(x)$  to the beam stiffness

Numerical modelling was performed in MATLAB to simulate the dynamic behaviour of a cantilever beam under inclined conditions. The material used in this study was steel, with a total beam length of 10 m, an elastic modulus of 210 GPa, and a density of 7850 kg/m<sup>3</sup>. The beam had a cross-sectional area of  $22.8 \times 10^{-4}$  m<sup>2</sup> and a second moment of area of  $935 \times 10^{-8}$  m<sup>4</sup>, shown in Figure 3. The structure was modelled in an inclined position at 60° relative to the horizontal plane. This configuration generated an axial compressive force due to the self-weight component acting under the gravitational acceleration ( $g = 9.81$  m/s<sup>2</sup>), along the beam axis. This condition led to a non-uniform axial force distribution along the beam, thereby affecting the geometric stiffness and the system's dynamic response.

```

1      clc; clear; close all;
2
3      %%
4      %% 1. PARAMETER FISIK BALOK
5      %%
6      L = 10;
7      n = 100;
8      E = 210e9;
9      I = 935e-8;
10     A = 22.8e-4;
11     rho = 7850;
12
13     theta = 60*pi/180;
14     g = 9.81;
15
16     Le = L/n;
17     m_linear = rho*A;

```

Figure 3. Initialization of the physical and geometric parameters of the cantilever beam

The frequency spectrum obtained from the simulation is presented in Figure 4. The spectrum was derived from the PSD of the free-vibration response of the beam tip, computed via a Fast Fourier Transform (FFT) of the transverse displacement time history. The simulation assumes free-

vibration conditions, initiated by releasing the beam from its undeformed position toward its static equilibrium under gravity, with a damping ratio  $\zeta = 0.02$ , consistent with the accepted value for structural steel. Two natural frequencies are identified from the spectrum: 1.844 Hz and 11.607 Hz. The first peak is significantly more prominent than the second. This can be physically explained by two factors: first, the applied initial condition primarily excites the fundamental mode; second, higher modes dissipate energy more rapidly because modal damping is proportional to  $\omega_n$ , so the energy of mode 2 dissipates before the PSD is computed. The system's vibration response is therefore predominantly governed by the fundamental mode.

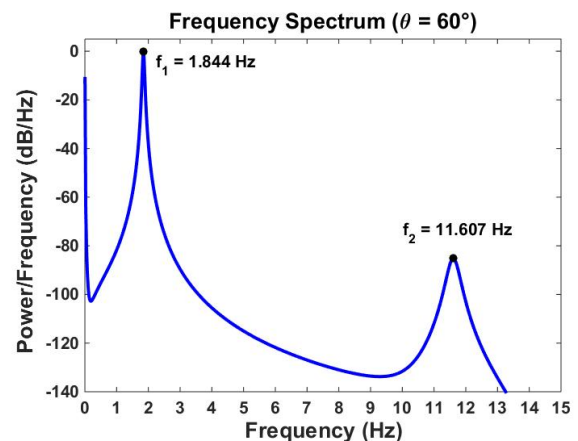
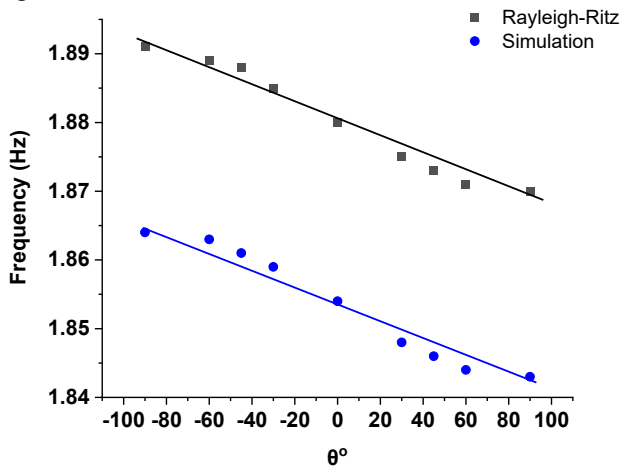


Figure 4. Frequency spectrum derived from the transverse free-vibration response of the inclined cantilever beam at  $\theta=60^\circ$ , showing the first and second natural frequencies at 1.844 Hz and 11.607 Hz, respectively

From a physical perspective, the observed characteristics are strongly governed by the inclusion of the geometric stiffness matrix ( $K_g$ ), which captures the second-order interaction between the axial compressive force and the beam's transverse deformation, as shown in Figure 2. In this study, the total structural stiffness is formulated as  $K_{total} = K_{elastic} + K_g$ , where  $K_g$  carries negative contributions arising from the compressive axial force derived from the beam self-weight (gravity). This force varies linearly along the beam — maximum at the fixed end and zero at the free end — and reduces the effective lateral stiffness, a phenomenon known as geometric softening. Without  $K_g$  The model would predict the natural frequency of an idealized beam under no axial load, which, from the Rayleigh–Ritz estimate, is 1.88 Hz at  $\theta = 0^\circ$ . When the geometric stiffness matrix  $K_g$  is included, the natural frequency decreases to 1.844 Hz for an inclination angle of  $\theta = 60^\circ$ . This reduction demonstrates that  $K_g$  Indeed, it plays a role in lowering the natural frequency, and the developed FEM model has successfully captured this effect numerically.

The obtained first natural frequency validated through comparison with the Rayleigh–Ritz analytical method using the polynomial shape function, which satisfies the geometric boundary conditions of a cantilever beam (zero deflection and zero slope at the fixed end). This shape function yields 1.871 Hz for  $\theta = 60^\circ$ , with a relative difference of approximately 1.47% from the FEM result.

This small overestimation is consistent with the upper-bound property of the Rayleigh-Ritz method: since the assumed shape function approximates the true mode shape, the method always yields a frequency equal to or greater than the exact solution. The good agreement between both methods confirms the validity of the present FEM model in capturing the dynamic behavior of the inclined cantilever beam. To further assess this agreement across a broader range of conditions, the inclination angle  $\theta$  was varied from  $-90^\circ$  to  $+90^\circ$ , with results summarized in Table 1 and illustrated in Figure 5.



**Figure 5.** Relationship between the inclination angle ( $\theta$ ) and the first natural frequency of the cantilever beam obtained using the Rayleigh–Ritz method and numerical simulation.

**Table 1.** Comparison of fundamental natural frequency  $f_1$  between Rayleigh–Ritz and FEM MATLAB for varying inclination angle  $\theta$

$\theta^\circ$	Rayleigh-Ritz (Hz)	FEM (Hz)	Error (%)
-90	1.891	1.863	1.456
-60	1.889	1.862	1.457
-45	1.888	1.860	1.459
-30	1.885	1.858	1.460
0	1.880	1.853	1.465
30	1.875	1.848	1.470
45	1.8731	1.846	1.472
60	1.871	1.844	1.473
90	1.870	1.842	1.475

Figure 5 presents the variation of the fundamental natural frequency as a function of inclination angle  $\theta$  for both methods. As shown in the figure, the two curves run nearly parallel over the entire range from  $-90^\circ$  to  $+90^\circ$ , with the Rayleigh–Ritz results consistently above the FEM results. This visual pattern is quantitatively confirmed by Table 1, which shows a consistent relative difference of approximately 1.47% across all angles. Both methods exhibit a clear decreasing trend in natural frequency with increasing  $\theta$ , which is physically governed by the  $\sin(\theta)$ -dependent compressive axial force  $N(x)$ , progressively reducing the effective lateral stiffness through the geometric softening effect. The fact that the offset between the two methods remains nearly constant across all inclination angles further confirms that both approaches capture the same underlying physical mechanism [29][27]. The consistent agreement between the analytical and numerical results validates the

present FEM model across the full range of inclination angles investigated

Despite the good agreement between the Rayleigh–Ritz solution and the present FEM model, the results remain bounded by the assumptions of the Euler–Bernoulli beam theory. In particular, shear deformation and rotary inertia are neglected, so the model is strictly valid for slender beams and low-to-moderate frequency ranges; for stockier beams, higher modes, or thicker cross-sections, refined beam theories such as Rayleigh or Timoshenko would be more appropriate and can noticeably alter the predicted eigen frequencies [30]–[32]. The present formulation is also based on small transverse deflections about a pre-buckled configuration, with geometric nonlinearity incorporated only through the geometric stiffness matrix  $K_g$ . At very large amplitudes, extreme rotations, or near-buckling states, geometrically exact formulations have been shown to be necessary to accurately capture hardening/softening behavior and mode coupling[33].Furthermore, material and viscous damping are represented by a constant modal damping ratio, whereas experimental and numerical studies indicate that damping can depend on amplitude, frequency, and boundary/interface conditions [34].

Finally, the influence of beam slenderness and inclination is examined here for a single geometry; varying the slenderness ratio, cross-sectional properties, or adding tip masses and elastic supports can significantly modify the gravity-induced softening trends reported in this work [35], [36]. Within these limitations, the present Euler–Bernoulli-based FEM model provides an efficient and sufficiently accurate tool for capturing the fundamental frequency variation with inclination for slender, lightly damped cantilever beams dominated by their first bending mode..

## Conclusion

This study confirms that the dynamic behavior of an inclined cantilever beam is significantly governed by the interaction between gravitational forces and structural orientation. The beam inclination generates a spatially varying axial compressive force, which induces geometric softening and reduces the effective lateral stiffness. Numerical results for a slender steel cantilever show that the natural frequencies decrease as the inclination angle increases. The close agreement between the Rayleigh–Ritz analytical solution and the spectral analysis of simulated transient responses supports the validity of the proposed model for slender, self-weight-dominated beams in the small-deflection regime. From an engineering perspective, these findings are most critical for long, slender cantilevered components such as masts, antenna booms, sign gantries, and inclined bridge elements, where self-weight contributes a significant share of the axial load, and the first bending mode governs serviceability or vibration-control criteria. Future work should extend the present framework by incorporating Timoshenko beam effects to account for shear deformation and rotary inertia in stockier members, modelling damping more realistically, and treating large-deflection geometric nonlinearities beyond the linear  $K_g$  formulation. Experimental validation of scaled inclined cantilever specimens and comparison with advanced nonlinear and Timoshenko-based models reported in the recent literature would further delineate the validity range of the current

assumptions and support safer modelling and design of real inclined cantilever structures.

### Author's Contribution

A.K.Y. Pratama: The author solely conceived and designed the study, developed the theoretical formulation, performed the numerical simulations, analyzed and interpreted the results, and wrote. Sunarko: reviewed and approved the final manuscript.

### References

- [1] R. Ali, M. Beh, A. Y. Al-dweik, B. S. Yilbas, and M. Sunar, "Heliyon Closed form solution of nonlinear oscillation of a cantilever beam using  $\lambda$ -symmetry linearization criteria," *Heliyon*, vol. 8, no. 11, p. e11673, 2022, doi: 10.1016/j.heliyon.2022.e11673.
- [2] A. Janchikoski and J. F. Meireles, "Dynamic Analysis of a Free Vibrating Cantilever Beam," *Rev. Científica Multidiscip. Núcleo do Conhecimento*, pp. 123–146, 2023.
- [3] P. J. Waste *et al.*, "Determination of Natural Frequencies and Mode Shapes of a Cantilever Beam using Finite Element Method," *Int. J. Sci. Res. Eng. Manag.*, vol. 09, no. 07, pp. 9–11, 2025, doi: 10.55041/IJSREM51691.
- [4] M. R. Chiarelli, "Dynamic Response of Beams Under Random Loads," *Mathematics*, vol. 13, 2025.
- [5] A. Moghadam, H. G. Melhem, and A. Esmaily, "A proof-of-concept study on a proposed ambient-vibration-based approach to extract pseudo-free-vibration response," *Eng. Struct.*, vol. 212, no. July 2019, p. 110517, 2020, doi: 10.1016/j.engstruct.2020.110517.
- [6] Y. Madhuranthakam and S. K. Chakrapani, "Analytical Study of Nonlinear Flexural Vibration of a Beam with Geometric, Material and Combined Nonlinearities," *Vibration*, vol. 7, pp. 464–478, 2024.
- [7] G. Shukla and Rajbahadur, "Numerical and Analytical Investigations of Vibration and Buckling in Composite Cantilever Beams," *J. Inf. Syst. Eng. Manag.*, vol. 10, 2025.
- [8] L. Li, "Natural Vibration Characteristics of a Cantilever Beam Parametrically Excited by Pitch Motion Pitch motion is one of the basic motions of insects wings. When the pith motion occurs, the wing rotates around a spanwise axis which can generate special flow," *Int. J. Struct. Stab. Dyn.*, vol. 24, no. 12, pp. 1–41, 2024, doi: 10.1142/S021945542550124X.
- [9] Z. Shi, C. Wang, and G. Yao, "Communications in Nonlinear Science and Numerical Simulation Stability and nonlinear vibrations of an inclined axially moving beam considering self-weight," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 133, no. January, p. 107966, 2024, doi: 10.1016/j.cnsns.2024.107966.
- [10] X. Fu, S. Wei, and H. Ding, "Weight Effects on Vertical Transverse Vibration of a Beam with a Nonlinear Energy Sink," *Appl. Sci.*, vol. 15, 2025.
- [11] D. Gritsenko, J. Xu, and R. Paoli, "Transverse vibrations of cantilever beams: Analytical solutions with general steady-state forcing," *Appl. Eng. Sci.*, vol. 3, no. September, 2020, doi: 10.1016/j.apples.2020.100017.
- [12] G. Wang, H. Ding, and L. Chen, "Dynamic effect of internal resonance caused by gravity on the nonlinear vibration of vertical cantilever beams," *J. Sound Vib.*, vol. 474, p. 115265, 2020, doi: 10.1016/j.jsv.2020.115265.
- [13] C. Iandiorio and P. Salvini, "Large displacements of slender beams in plane: Analytical solution by means of a new hypergeometric function," *Int. J. Solids Struct.*, vol. 185–186, pp. 467–484, 2020, doi: 10.1016/j.ijsolstr.2019.09.006.
- [14] V. Ondra and B. Titurus, "Free vibration and stability analysis of a cantilever beam axially loaded by an intermittently attached tendon," *Mech. Syst. Signal Process.*, vol. 158, p. 107739, 2021, doi: 10.1016/j.ymsp.2021.107739.
- [15] G. Wang, H. Ding, and L. Chen, "Gravitational effects and mode interactions of vertical cantilever beams," *Int. J. Non. Linear. Mech.*, vol. 123, no. April, p. 103493, 2020, doi: 10.1016/j.ijnonlinmec.2020.103493.
- [16] K. Kumar and S. S. Gupta, "International Journal of Solids and Structures Planar nonlinear dynamics of a hanging geometrically exact beam with a moving mass," *Int. J. Solids Struct.*, vol. 325, no. September 2025, p. 113704, 2026, doi: 10.1016/j.ijsolstr.2025.113704.
- [17] H. S. A. Ebrahimi-mamaghani and M. Safarpour, "Nonlocal study of the vibration and stability response of small-scale axially moving supported beams on viscoelastic-Pasternak foundation in a hygro-thermal environment," *Math. Methods Appl. Sci.*, no. July, pp. 1–12, 2020, doi: 10.1002/mma.6859.
- [18] Y. Liu, "Nonlinear Dynamic Analysis of an Axially Moving Composite Laminated Cantilever Beam," *J. Vib. Eng. Technol.*, no. 0123456789, 2022, doi: 10.1007/s42417-022-00750-2.
- [19] X. Liu, C. Sun, J. R. Banerjee, H. Dan, and L. Chang, "An exact dynamic stiffness method for multibody systems consisting of beams and rigid-bodies," *Mech. Syst. Signal Process.*, vol. 150, p. 107264, 2021, doi: 10.1016/j.ymsp.2020.107264.
- [20] J. R. Banerjee, "An exact method for free vibration of beams and frameworks using frequency-dependent mass, elastic and geometric stiffness matrices," *Comput. Struct.*, vol. 292, no. August 2023, p. 107235, 2024, doi: 10.1016/j.compstruc.2023.107235.
- [21] I. Christiawan, "Secara umum kompromi," p. 10, 1998.
- [22] G. Nitti, G. Lacidogna, and A. Carpinteri, "An analytical formulation to evaluate natural frequencies and mode shapes of high-rise buildings," *Curved and Layer. Struct.*, vol. 8, pp. 307–317, 2021.
- [23] L. Ledda, A. Greco, I. Fiore, and I. Calio, "Closed-Form Exact Solution for Free Vibration Analysis of Symmetric Functionally Graded Beams," *Symmetry (Basel)*, vol. 16, 2024.
- [24] M. S. Sari and S. Faroughi, "Free vibrations of functionally graded porous hanging and standing cantilever beams," *ournal Low Freq. Noise, Vib. Act.*

- Control*, vol. 43, no. 3, pp. 1082–1096, 2024, doi: 10.1177/14613484241238593.
- [25] Z. Shi, C. Wang, and G. Yao, “Stability and nonlinear vibrations of an inclined axially moving beam considering self-weight,” *Commun. Nonlinear Sci. Numer. Simul.*, vol. 133, no. January, p. 107966, 2024, doi: 10.1016/j.cnsns.2024.107966.
- [26] M. Soltani and B. Asgarian, “New hybrid approach for free vibration and stability analyses of axially functionally graded Euler-Bernoulli beams with variable cross-section resting on uniform Winkler-Pasternak foundation,” *Lat. Am. J. Solids Struct.*, vol. 16, no. 3, pp. 1–25, 2019.
- [27] M. Debeurre, “Extreme nonlinear dynamics of cantilever beams : effect of gravity and slenderness on the nonlinear modes,” *Nonlinear Dyn.*, vol. 111, no. 14, pp. 12787–12815, 2023, doi: 10.1007/s11071-023-08637-x.
- [28] M. A. Eltahir, A. E. Alshorbagy, and F. F. Mahmoud, “Vibration analysis of Euler – Bernoulli nanobeams by using finite element method,” *Appl. Math. Model.*, vol. 37, no. 7, pp. 4787–4797, 2013, doi: 10.1016/j.apm.2012.10.016.
- [29] Q. Abbas, R. Nawaz, H. Yaqoob, H. Muhammad, and M. Musaddiq, “Nonlinear vibration analysis of cantilever beams : Analytical , numerical and experimental perspectives,” *Partial Differ. Equations Appl. Math.*, vol. 13, no. December 2024, p. 101115, 2025, doi: 10.1016/j.padiff.2025.101115.
- [30] Y. Chen, “Error Analysis Based on Euler-Bernoulli Beam Theory : An Example of a Simply Supported Beam Bending Model,” in *CONF-FMCE 2025 Symposium: Semantic Communication for Media Compression and Transmission*, 2025, vol. 0, pp. 17–24. doi: 10.54254/2755-2721/155/2025.GL23166.
- [31] A. Luo, B. Lossouarn, and A. Erturk, “The effects of shear deformation and rotary inertia on the electrical analogs of beams and plates for multimodal piezoelectric damping,” *Int. J. of Circuit Theory Appl.*, vol. 52, no. September 2023, pp. 2985–2998, 2024, doi: 10.1002/cta.3899.
- [32] G. Kanwal, N. Ahmed, and R. Nawaz, “A comparative analysis of the vibrational behavior of various beam models with different foundation designs,” *Heliyon*, vol. 10, no. 5, p. e26491, 2024, doi: 10.1016/j.heliyon.2024.e26491.
- [33] H. Farokhi, E. Kohtanen, and A. Erturk, “Extreme parametric resonance oscillations of a cantilever : An exact theory and experimental validation,” *Mech. Syst. Signal Process.*, vol. 196, no. April, p. 110342, 2023, doi: 10.1016/j.ymsp.2023.110342.
- [34] H. Farokhi, Y. Xia, and A. Erturk, “Experimentally validated geometrically exact model for extreme nonlinear motions of cantilevers,” *Nonlinear Dyn.*, vol. 107, no. 1, pp. 457–475, 2022, doi: 10.1007/s11071-021-07023-9.
- [35] F. H. Jazi, R. Tikani, and M. S. Taki, “Geometric modeling and dynamic analysis of cantilever beams for,” *Multidiscip. Model. Mater. Struct.*, vol. 21, no. 4, pp. 916–941, 2025, doi: 10.1108/MMMS-10-2024-0301.
- [36] M. A. Mahmoud, “Natural frequency of axially functionally graded , tapered cantilever beams with tip masses,” *Eng. Struct.*, vol. 187, no. February, pp. 34–42, 2019, doi: 10.1016/j.engstruct.2019.02.043.