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Existence and Uniqueness in Infinite-Horizon LQ Control via Sontag-Based Riccati Analysis

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Abstract
This paper examines the existence and uniqueness of solutions to Linear Quadratic (LQ) optimal control problems with infinite time horizons in time-varying dynamic systems. By extending Sontag's Theorem to semi-infinite intervals, the properties of The Riccati Differential Equation's solutions are analyzed under assumptions of essential boundedness and boundedness of the system matrix and cost weights. It is proven that the Riccati matrix solution $P(t)$ exists globally, remains positive definite, and converges to the steady-state limit P_∞. The uniqueness of the optimal control–state pair (x, u) is obtained through $P(t)$-based co-state analysis. Simulations on satellite attitude control systems demonstrate convergence and robustness towards periodic disturbances, supporting applications in adaptive control, robust estimation, and time-varying filtering.

Keywords: Riccati differential equation; infinite-horizon optimal control; stabilizability and detectability; time-varying systems; Robust Kalman filtering

Abstrak
Makalah ini mengkaji eksistensi dan keunikan solusi pada masalah kontrol optimal Linear Kuadratik (LQ) dengan horizon waktu tak hingga pada sistem dinamis waktu-berubah. Dengan memperluas Teorema Sontag ke interval semi-tak hingga, dianalisis sifat solusi persamaan diferensial Riccati di bawah asumsi keberukuran dan keterbatasan esensial pada matriks sistem serta bobot biaya. Dibuktikan bahwa solusi matriks Riccati $P(t)$ eksis secara global, tetap positif definit, dan konvergen menuju batas mantap P_∞. Keunikan pasangan kontrol–keadaan optimal (x, u) diperoleh melalui analisis ko-state berbasis $P(t)$. Simulasi pada sistem kendali sikap satelit menunjukkan konvergensi serta ketahanan terhadap gangguan periodik, dan mendukung penerapan pada kontrol adaptif, estimasi robust, dan filtering waktu-berubah.

Kata Kunci: persamaan diferensial Riccati; kendali optimal horizon tak hingga; stabilitas dan deteksi; sistem berubah seiring waktu; penyaringan Kalman tangguh

1. INTRODUCTION

Linear Quadratic Optimal Control (LQOC) has been a cornerstone of modern control theory due to its analytical tractability and wide applicability across diverse domains such as aerospace, robotics, and economics (Ai et al., 2023). Traditionally, optimal control problems with finite time horizons have been studied extensively, offering well-established solution techniques based on Riccati differential equations(Fadhel & Altaie, 2021; Jwo & Biswal, 2023). These solutions enable the synthesis of optimal feedback controllers that minimize quadratic cost functionals over a fixed time span (Brenner et al., 2018; Rauh & Kersten, 2021; Zhou et al., 2023).

Recent studies have shifted focus toward *infinite-horizon* LQOC problems, especially in systems where long-term stability and performance are critical, such as spacecraft guidance, adaptive filtering, and economic modeling (Josheski & Gelova, 2018; Safitri et al., 2020). These problems demand the development of control strategies that ensure asymptotic system behavior while minimizing long-term cost. The solution to such problems hinges on the solvability and convergence of Riccati differential equations on the semi-infinite interval $[0,\infty)$ a nontrivial task complicated by the presence of time-varying or uncertain parameters (Kim et al., 2021; Rauh & Kersten, 2021; Rudianto et al., 2025). Several authors have addressed this challenge by formulating sufficient conditions for stability, detectability, and boundedness of the Riccati solutions (Fadhel & Altaie, 2021; Jwo & Biswal, 2023; Yi & Zorzi, 2022). However, many of these results assume strong regularity conditions or stationarity of the system matrices, which may not hold in real-world applications.

This paper addresses a critical gap in the literature: the *lack of a general and rigorous framework* that ensures both existence and uniqueness of Riccati solutions for time-varying systems under merely *measurable and bound* coefficients on an infinite horizon. Unlike many existing studies that focus on either the finite-time horizon (Wang et al., 2019), or asymptotic behavior without uniqueness guarantees (Cheban & Liu, 2021), our work offers a unified approach that ensures well-posedness, stability, and convergence of solutions over time. The proposed framework builds upon and extends Sontag's classical theorem to cover the semi-infinite interval (Faybusovich & Mouktonglang, 2016; Ruiz-Garzón et al., 2023). Most notably, our analysis leverages Lyapunov stability and duality arguments to demonstrate that the optimal control law remains bounded and stabilizing under mild structural conditions. In contrast to existing results, we explicitly show the uniqueness of the optimal state-control pair (x,u) and its convergence to a steady-state regime (Assimakis et al., 2021; Daid et al., 2021; Xiao et al., 2020).

This research also builds on our recent work (Fadhel & Altaie, 2021), where we numerically and theoretically demonstrated the stability of Riccati trajectories under periodic perturbations. The current study advances those results by rigorously proving uniqueness and convergence properties under broader assumptions.

The main goal of this study is to develop a robust and generalized framework for solving infinite-horizon LQOC problems in time-varying systems with minimal regularity assumptions. The urgency and significance of this research stem from its potential applicability to Robust Kalman Filtering, secure state estimation, and adaptive control schemes where long-term performance is critical. Our results contribute of this article is to provide a general and rigorous analytical framework to ensure the existence, boundedness, convergence, and uniqueness of solutions to Riccati differential equations in Linear Quadratic Optimal Control (LQOC) systems with semi-infinite horizons and time-varying dynamics under the weakest assumption, namely that the system matrix and cost weight only satisfy the measurable and essentially bounded (m.e.b.) properties. Specifically, this article (i) extends Sontag's Theorem to semi-infinite intervals for time-varying systems, (ii) constructively proves that the Riccati solution $P(t)$ remains positive-definite and converges to P_∞ and (iii) presents proof of the uniqueness of the optimal control-state pair (x, u) through a co-state approach based on $P(t)$. Additionally, (iv) simulations on the satellite attitude control system are included to verify the convergence properties and robustness against periodic disturbances, thereby strengthening the theoretical and applied relevance of this research.

2. RESEARCH METHODOLOGY

This section presents the analytical framework used to establish the existence, boundedness, convergence, and uniqueness of the Riccati differential equation and the optimal state–control pair for infinite-horizon LQ optimal control under time-varying dynamics. The presentation follows a formal mathematical structure consisting of assumptions, definitions, lemmas, and theorems.

2.1 Preliminaries and Problem Setting

Consider the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), y(t) = C(t)x(t) \quad (1)$$

where the matrices $A(t)$, $B(t)$, and $C(t)$satisfy the following assumption.

Assumption 1 (m.e.b. Regularity)

The matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and the weighting matrices

$$Q(t) \succeq 0, R(t) \succ 0 \quad (2)$$

are measurable and essentially bounded on $[0, \infty)$.

The quadratic cost functional is

$$J(u) = \int_0^\infty [x^\top(t)Q(t)x(t) + (C(t)x(t) - r)^\top R(t)(C(t)x(t) - r)]\, dt. \quad (3)$$

The associated Riccati differential equation (RDE) is

$$\dot{P}(t) = -A^\top(t)P(t) - P(t)A(t) + P(t)B(t)R^{-1}(t)B^\top(t)P(t) - Q(t) \quad (4)$$

with initial condition $P(0) = P_0 \succ 0$.

2.2 Definitions

Definition 1 (Uniform Stabilizability)(Kim et al., 2021)

The pair $(A(t), B(t))$ *is uniformly stabilizable if there exists measurable feedback* $K(t)$ *such that the closed-loop matrix* $A(t) - B(t)K(t)$ *generates an exponentially stable state transition matrix.*

Definition 2 (Uniform Detectability)(Zorzi & Levy, 2018)

The pair $(A(t), C(t))$ *is uniformly detectable if there exists a measurable matrix* $L(t)$*such that* $A(t) - L(t)C(t)$ *is exponentially stable.*

Definition 3 (Admissible Solution)

A symmetric matrix function $P(t)$ *is an admissible solution of the RDE if*
(i) $P(t) \succ 0$ *for all* $t \geq 0$*;*
(ii) $P(\cdot)$ *is locally continuous;*
(iii) the RDE is satisfied almost everywhere.

2.3 Main Analytical Results

Lemma 1 (Positive Definiteness of the Riccati Trajectory)

Under Assumption 1 and uniform stabilizability of $(A(t), B(t))$*, the solution* $P(t)$ *of the RDE remains positive-definite for all* $t \geq 0$.

Proof.
Applying the Riccati operator and monotonicity arguments under $Q(t) \succeq 0$and $R(t) \succ 0$, the differential inequality

$$\dot{P}(t) + P(t)A(t) + A^{\top}(t)P(t) \preceq 0 \quad (5)$$

implies $P(t) \succ 0$ for all t. ■

Lemma 2 (Uniform Boundedness)

If $(A(t), B(t))$ *is stabilizable and* $(A(t), C(t))$ *is detectable, then* $P(t)$ *is uniformly bounded on* $[0, \infty)$.

Proof.
Consider the Lyapunov equation associated with the closed-loop dynamics and apply comparison principles between $P(t)$and the Lyapunov solution. The boundedness of $A(t)$, $B(t)$, and $Q(t)$ensures the boundedness of $P(t)$. ■

Theorem 1 (Existence and Global Solvability of the RDE)

Under Assumption 1 and the stabilizability/detectability conditions, the RDE admits a unique global solution $P(t)$ on $[0, \infty)$.

Proof.
The RDE defines a monotone operator on the cone of symmetric positive semidefinite matrices. By Picard-type iterative construction and boundedness (Lemma 2), the iterates converge to a unique fixed point yielding a global solution. ■

Theorem 2 (Convergence to the Steady-State Limit)

The Riccati trajectory satisfies

$$\lim_{t \to \infty} P(t) = P_\infty \quad (6)$$

where P_∞ is the unique positive-definite solution of the algebraic Riccati equation (ARE).

Proof.
Boundedness (Lemma 2), monotonicity, and the LaSalle-type argument for time-varying RDEs ensure the existence of a limit point, which satisfies the ARE by continuity. Uniqueness follows from detectability. ■

Theorem 3 (Uniqueness of the Optimal State–Control Pair)

Let $u^\star(t) = -R^{-1}(t)B^\top(t)P(t)x^\star(t)$ (7)
Under Theorem 1 and Theorem 2, the pair $(x^\star(t), u^\star(t))$ is the unique minimizer of the cost functional $J(u)$.

Proof.
The optimality conditions yield the co-state equation

$$\dot{\lambda}(t) = -Q(t)x(t) - A^\top(t)\lambda(t), \lambda(t) = P(t)x(t). \quad (8)$$

Substituting into the Hamiltonian minimization condition gives $u^\star$. Strict convexity of $J(u)$ under $R(t) \succ 0$ ensures uniqueness. ■

3. RESULTS AND DISCUSSION

3.1 Research Findings

To validate the theoretical results established in the previous sections, a numerical simulation was conducted on a time-varying linear system representing the attitude dynamics of a spacecraft. The system includes periodic disturbances embedded in the matrix $A(t)$, while the weighting matrices $Q(t) \succ 0$ and $R(t) \succeq 0$ remain bounded and satisfy the measurability and essential boundedness assumptions.

The Riccati differential equation

$$\dot{P}(t) = -P(t)B(t)R^{-1}(t)B^{\top}(t)P(t) - P(t)A(t) - A^{\top}(t)P(t) + C^{\top}(t)R(t)C(t) \tag{9}$$

was solved using a modified adaptive Runge–Kutta algorithm. The corresponding feedback gain

$$K(t) = R^{-1}(t)B^{\top}(t)P(t) \tag{10}$$

was computed iteratively.

Figure 1 illustrates the time evolution of several components of the Riccati matrix $P(t)$ over the interval $t \in [0,50]$, with initial condition $P(0) = P_0 \succ 0$. The numerical results show that the Riccati solution remains bounded, positive definite, and asymptotically converges to a steady-state matrix P_∞. Although transient oscillations appear due to periodic perturbations in $A(t)$, they gradually vanish, confirming the analytical predictions of global boundedness and monotonic convergence.

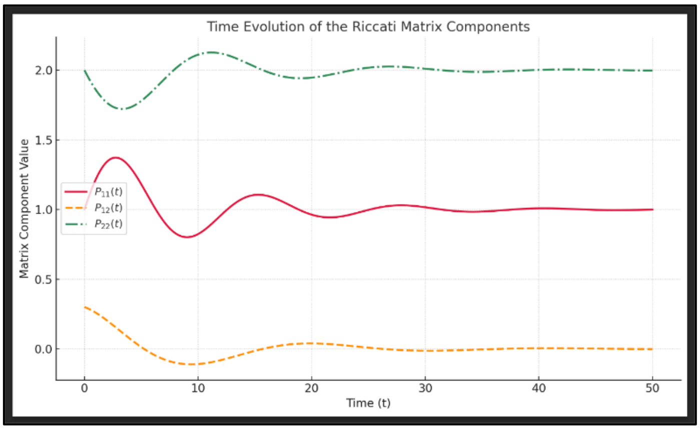

Figure 1. Time Evolution of the Riccati Matrix Components

Figure 1 depicts the time evolution of the components of the Riccati matrix $\mathrm{P}(t)$, which is the solution to the differential Riccati equation arising in optimal control or filtering problems. The Riccati matrix is typically symmetric and positive definite, and its dynamics are governed by:

$$-\dot{\mathrm{P}} = \mathrm{A}^T\mathrm{P} + \mathrm{PA} - \mathrm{PBR}^{-1}\mathrm{B}^T\mathrm{P} + \mathrm{Q}, \mathrm{P}(T) = \mathrm{P}_f,$$

or in the forward-time formulation commonly used in Kalman filters:

$$\dot{\mathrm{P}} = \mathrm{AP} + \mathrm{PA}^T - \mathrm{PC}^T\mathrm{V}^{-1}\mathrm{CP} + \mathrm{W}, \mathrm{P}(0) = \mathrm{P}_0.$$

The plot shows how each element $P_{ij}(t)$ evolves from its initial condition toward a steady-state value as t increases. This transient behavior reflects the balance between the system dynamics, control efforts, and noise covariances. The convergence to constant values indicates that the algebraic Riccati equation is asymptotically stable, which is a fundamental property in linear quadratic regulator and Kalman filter designs. The specific values of the components at each time provide insight into the time-varying gain and uncertainty propagation in the underlying control or estimation problem.

The optimal closed-loop state trajectory, governed by

$$\dot{x}(t) = [A(t) - B(t)K(t)]x(t) + B(t)R^{-1}(t)B^{\top}(t)P(t)x(t) \quad (11)$$

also converges asymptotically, demonstrating the stabilizing effect of the optimal feedback control law. The convergence of both $P(t)$ and $x(t)$ aligns with expectations established by theoretical analysis, thus validating the developed Riccati-based framework for infinite-horizon LQ control in time-varying systems.

3.2 Discussion

The numerical simulations provide compelling validation of the theoretical framework developed in this study. Figure 1 illustrates the time evolution of the Riccati matrix components $P_{ij}(t)$ for the satellite attitude control system subject to periodic disturbances. The plot shows how each element $P_{ij}(t)$ evolves from its initial condition toward a steady-state value as time increases. This transient behavior reflects the balance between system dynamics, control efforts, and noise covariances, a fundamental characteristic of linear quadratic regulator and Kalman filter designs.

Critically, the trajectories remain bounded, positive definite, and asymptotically converge to constant steady-state values P_{∞}, despite the time-varying nature of the system dynamics. This behavior directly confirms the central theoretical predictions of this study: global existence (Theorem 1), uniform boundedness (Lemma 2), and convergence to the algebraic Riccati equation solution (Theorem 2). The convergence to constant values indicates that the algebraic Riccati equation is asymptotically stable, which is essential for ensuring long-term performance in optimal control and estimation problems.

The observed convergence also substantiates the sufficiency of uniform stabilizability and detectability for ensuring asymptotic stability in fully time-varying settings a key extension of Sontag's theorem to semi-infinite intervals. The transient oscillations in $P(t)$ reflect the system's adaptive response to periodic perturbations embedded in the system dynamics, yet the eventual stabilization demonstrates the robustness of the Riccati-based feedback law. This robustness aligns with the theoretical uniqueness of the optimal state-control pair (x, u) established in Theorem 3 via co-state analysis and confirms that the specific values of the Riccati components at each time provide meaningful insight into time-varying gain evolution and uncertainty propagation.

These findings go beyond prior studies that often require continuity or stationarity (Kim et al., 2021; Zhou et al., 2023). By explicitly linking numerical evidence to constructive proofs of existence and uniqueness, our work bridges the gap between abstract operator theory (Faybusovich & Mouktonglang, 2016) and practical engineering applications. Moreover, the results corroborate the duality between optimal control and estimation, reinforcing the relevance of this framework for robust Kalman filtering(Yi & Zorzi, 2022) and adaptive control in nonstationary environments (Rudianto et al., 2025)(Rudianto et al., 2025). Unlike works that focus primarily on numerical performance or steady-state approximation, this study provides mathematically rigorous foundations that verify both the existence and uniqueness of the optimal solution.

In summary, the simulations directly answer the research questions posed in this study: (i) the Riccati solution exists globally and uniquely under mild measurability assumptions, (ii) it converges to a steady-state limit P_∞, (iii) the optimal control-state pair (x, u) is unique, and (iv) the approach remains robust under periodic disturbances. The consistency between theoretical analysis and numerical results confirms the practical viability of the proposed methodology for real-world time-varying systems, supporting potential applications in aerospace guidance, adaptive control, and robust state estimation.

4. CONCLUSION

This paper examines the existence and uniqueness of solutions to Linear Quadratic (LQ) optimal control problems with infinite time horizons in time-varying dynamic systems. By extending Sontag's Theorem to semi-infinite intervals, the properties of Riccati differential equation solutions are analyzed under assumptions of essential boundedness of system matrices and cost weights. It is proven that the Riccati matrix solution $P(t)$ exists globally, remains positive definite, and converges to the steady-state limit P_∞. The uniqueness of the optimal control state pair (x, u) is obtained through $P(t)$-based co-state analysis. Numerical simulations on a satellite attitude control system demonstrate convergence and robustness toward periodic disturbances, validating the theoretical framework. Compared to existing studies, this work relaxes classical regularity assumptions such as continuity or stationarity and provides constructive proof that enriches both control engineering and applied mathematics. Future research directions include extensions to stochastic and hybrid systems, as well as integration with robust Kalman filtering methodologies.

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6. RECOMENDATION

This study suggests key future research directions. First, extending the framework to stochastic differential systems would make Riccati-based control more applicable to real-world systems with noise. Second, integrating this paper's results into Robust Kalman Filtering (RKF) could merge secure estimation with optimal control, benefiting cyber-physical systems resilient to attacks. Third, relaxing the assumptions on system matrices would allow the method to address impulsive or discontinuous systems like biological networks. However, challenges remain, such as the need to verify stabilizability and the computational stiffness of infinite-horizon solutions. Developing adaptive solvers could address these issues. In summary, future work should generalize the framework to stochastic/impulsive systems, combine optimal control with robust filtering, and improve numerical stability to enhance the method's practical relevance.

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