



Unified Riccati-Based Analysis for Robust Kalman Filter Design on Semi-Infinite Time Intervals

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Abstract

Robust state estimation remains challenging under model uncertainties and time-varying dynamics, as Kalman filtering degrades when models deviate from nominal representations. The Riccati equation governs optimal control and estimation, yet classical treatments impose limiting assumptions. This article extends Sontag's theorem to time-varying systems with measurable and essentially bounded coefficients, establishing a unified framework for robust Kalman filter design on semi-infinite intervals addressing existence, uniqueness, and convergence under minimal regularity. The framework formulates robust filtering as a minimax game and employs Riccati analysis to prove global existence, convergence to a steady-state limit, and uniqueness of the optimal pair, validated through satellite attitude control simulations. Results confirm global existence, asymptotic convergence, and stability against periodic disturbances. This paper extends Sontag's theorem on the existence and boundedness of solutions to Continuous-Time Algebraic Riccati Equations (CARE) from a finite interval $[0, T]$ to the semi-infinite interval $[0, \infty)$. The main novelty lies in establishing unified Riccati-based sufficient conditions that explicitly incorporate norm-bounded parameter uncertainties, thereby guaranteeing the asymptotic stability and H_∞ performance of the Robust Kalman Filter design over an infinite time horizon. However, the framework is limited to deterministic linear systems and excludes stochastic noise and nonlinear dynamics.

Keywords: riccati differential equation; robust kalman filter; semi-infinite interval; time-varying systems; optimal control

Abstrak

Estimasi keadaan tangguh masih menantang di bawah ketidakpastian model dan dinamika berubah waktu, karena filter Kalman menurun ketika model menyimpang dari representasi nominal. Persamaan Riccati mengatur kontrol optimal dan estimasi, namun perlakuan klasik menerapkan asumsi yang membatasi. Artikel ini memperluas teorema Sontag ke sistem berubah waktu dengan koefisien terukur dan terbatas esensial, membangun kerangka terpadu untuk desain filter Kalman tangguh pada interval semi-tak hingga yang menangani keberadaan, keunikan, dan konvergensi di bawah regularitas minimal. Kerangka merumuskan penyaringan tangguh sebagai permainan minimaks dan menggunakan analisis Riccati untuk membuktikan keberadaan global, konvergensi ke nilai tunak, dan keunikan pasangan optimal, divalidasi dengan simulasi kontrol sikap satelit. Hasil mengonfirmasi keberadaan global, konvergensi asimtotik, dan stabilitas terhadap gangguan periodik. Penelitian ini memperluas Teorema Sontag mengenai eksistensi dan kestabilan solusi Persamaan Riccati Kontinu dari interval terhingga $[0, T]$ ke interval setengah tak hingga $[0, \infty)$. Kebaruan utama terletak pada formulasi kondisi cukup

(sufficient conditions) yang mengintegrasikan batas ketidakpastian norm-bounded pada matriks sistem ke dalam struktur Riccati terpadu, sehingga menjamin kestabilan asimtotik dan performa H_∞ pada desain Filter Kalman Kekal (Robust Kalman Filter) untuk domain waktu tak-terbatas. Namun, kerangka ini terbatas pada sistem linear deterministik dan belum mencakup gangguan stokastik serta dinamika nonlinier.

Kata Kunci: persamaan diferensial riccati; robust kalman filter; interval semi-infinite; sistem berubah waktu; kontrol optimal

1. INTRODUCTION

The design of robust state estimators for dynamic systems operating over extended time horizons remains a fundamental challenge in control engineering and applied mathematics (Fang et al., 2022; Rauh & Kersten, 2021; Rudianto et al., 2026). Kalman filtering, despite its widespread success, exhibits performance degradation when the actual system model deviates from its nominal representation a condition frequently encountered in practical applications ranging from aerospace guidance to cyber-physical systems (Farahi & Yazdi, 2020; Hu et al., 2022; Rudianto et al., 2025b; C. Wang et al., 2020). This vulnerability has motivated extensive research into robust filtering methodologies that maintain estimation accuracy under model uncertainty (Fang et al., 2022; Li et al., 2022).

The Riccati differential equation occupies a central position in both optimal control and estimation theory, governing the evolution of the error covariance matrix in Kalman filters and the optimal cost-to-go in linear quadratic regulators (Jwo & Biswal, 2023; Rudianto et al., 2025a, 2026; Yi & Zorzi, 2022). However, classical treatments often impose restrictive assumptions on system regularity such as continuity or stationarity that limit their applicability to real-world time-varying systems (Peranginangin, 2023; Rudianto et al., 2025b). Recent advances have begun to relax these constraints, yet a unified framework that simultaneously addresses existence, uniqueness, and convergence of Riccati solutions on semi-infinite intervals remains elusive (Rauh & Kersten, 2021; Rudianto et al., 2026; Ruiz-Garzón et al., 2023; Z. Wang et al., 2023).

This article addresses this critical gap by developing a rigorous analytical framework for Riccati-based robust Kalman filter design under the weakest possible regularity assumptions. Specifically, we extend Sontag's theorem to semi-infinite intervals for time-varying systems characterized by measurable and essentially bounded (m.e.b.) coefficients. Our approach establishes: (i) global existence and positive definiteness of the Riccati solution $P(t)$, (ii) uniform boundedness and convergence to a steady-state limit P_∞ , and (iii) uniqueness of the optimal control-state pair (x, u) via co-state analysis based on $P(t)$.

The urgency and significance of this research stem from its potential to unify optimal control and robust state estimation under a common mathematical framework (Rudianto et al., 2025b; R. Wang et al., 2023; Yi & Zorzi, 2022). Recent work shows robust Kalman filters involve Riccati-like iterations on positive semidefinite matrices, with low-rank extensions for degenerate densities. Our constructive proofs provide theoretical foundations. Previous studies have extensively analyzed the existence of Continuous-

Time Algebraic Riccati Equation (CARE) solutions and Kalman filter stability. However, classical formulations, such as Sontag's theorem (1998), are predominantly constrained to finite time intervals $[0, T]$ and do not explicitly incorporate system parameter uncertainties over an infinite horizon $[0, \infty)$. To address this gap, the objective of this study is to establish unified Riccati-based sufficient conditions that extend Sontag's theorem to the semi-infinite interval $[0, \infty)$, thereby guaranteeing both asymptotic stability and H_∞ performance for Robust Kalman Filter design in the presence of norm-bounded parameter uncertainties.

2. RESEARCH METHODOLOGY

2.1. Preliminaries and Problem Setting

Consider the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t); y(t) = C(t)x(t) + v(t), \quad (1)$$

where $x(t) \in R^n$ denotes the attitude-related state vector, $u(t) \in R^m$ is the control input, and $y(t) \in R^p$ represents the measured output. The terms $w(t)$ and $v(t)$ denote the process and measurement disturbances, respectively. The system matrices $A(t)$, $B(t)$, and $G(t)$ are time-varying and periodic with a common period T_p , i.e.,

$$A(t + T_p) = A(t), \quad B(t + T_p) = B(t), \quad G(t + T_p) = G(t),$$

for all $t \geq 0$. In the present simulation, the period T_p is selected according to the benchmark configuration adopted from the prior work and the weighting matrices $Q(t) \succcurlyeq 0, R(t) \succ 0$ are measurable and essentially bounded on $[0, \infty)$.

The associated quadratic cost functional is

$$J(u) = \int_0^\infty [x^T(t)Q(t)x(t) + (C(t)x(t) - r)^T R(t)(C(t)x(t) - r)] dt. \quad (2)$$

The Riccati differential equation (RDE) governing the optimal control problem is

$$\dot{P}(t) = -A^T(t)P(t) - P(t)A(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) - Q(t). \quad (3)$$

To establish the main stability and filtering results, the following fundamental assumptions are imposed throughout this paper:

1. **Assumption 1 (Detectability and Stabilizability):** The nominal matrix pair (A, C) is detectable, and (A, B) is stabilizable on $[0, \infty)$.
2. **Assumption 2 (Norm-Bounded Uncertainties):** The uncertainty matrices satisfy $\Delta A(t) = D_1 F_1(t) E_1$ with $F_1^T(t) F_1(t) \leq I$ for all $t \geq 0$.
3. **Assumption 3 (Noise Covariances):** The process noise $w(t)$ and measurement noise $v(t)$ have positive definite covariance matrices $w(t) \succ 0$ and $v(t) \succ 0$ in $L_2[0, \infty)$.

2.2. Existence and Uniqueness Results

Theorem 1 (Global Existence and Convergence)

Under Assumption 1, the Riccati differential equation admits a unique solution $P(t)$ on $[0, \infty)$ that remains positive definite, uniformly bounded, and converges to a constant steady-state limit $P_\infty P_\infty$ satisfying the algebraic Riccati equation.

The proof relies on extending Sontag's theorem to the semi-infinite interval setting, demonstrating that the solution trajectory remains bounded despite the time-varying nature of system dynamics. The convergence to constant values substantiates the asymptotic stability of the algebraic Riccati equation, which is essential for ensuring long-term performance in optimal control and estimation problems (Rudianto et al., 2026).

Theorem 2 (Uniqueness of Optimal Pair)

The optimal control-state pair (x, u) is unique, as established through co-state analysis based on $P(t)$. This result confirms the specific values of the Riccati components at each time provide meaningful insight into time-varying gain evolution and uncertainty propagation (Rudianto et al., 2026).

2.3. Duality with Kalman Filtering

The duality between optimal control and estimation is central to our framework. The Kalman filter Riccati equation for state estimation error covariance $\Sigma(t)$ takes the dual form

$$\dot{\Sigma}(t) = A^T(t)\Sigma(t) + \Sigma(t)A(t) - \Sigma(t)C^T(t)R^{-1}(t)C(t)\Sigma(t) + B(t)Q(t)B^T(t). \quad (4)$$

The robust Kalman filter solves a minimax game: nature selects the least favorable model in a prescribed ambiguity set, while the other player designs the optimum filter for the least favorable model. This formulation yields a Riccati-like iteration evolving on the cone of positive semidefinite matrices, as established in recent work.

3. RESULT AND DISCUSSION**3.1 Simulation Setup**

To validate the theoretical framework, we consider a satellite attitude control system subject to periodic disturbances, following the simulation paradigm established in prior work. The system is described by the continuous-time linear time-varying model (1).

The time-varying system matrices used in the simulation are given by

$$\begin{aligned} A(t) &= \begin{bmatrix} 0 & 1 \\ -2.00 + 0.18 \sin(2\pi t/T_p) + 0.10 \cos(2\pi t/T_p) & -0.85 \end{bmatrix} \\ B(t) &= \begin{bmatrix} 0 \\ 1 + 0.08 \cos(2\pi t/T_p) \end{bmatrix}; C(t) = [1 \quad 0.12 \sin(2\pi t/T_p)] \\ Q(t) &= \begin{bmatrix} 4.00 + 0.30 \sin(2\pi t/T_p) & 0 \\ 0 & 1.00 + 0.10 \sin(2\pi t/T_p) \end{bmatrix} \\ R(t) &= [0.20 + 0.03 \cos(2\pi t/T_p)] \\ K(t) &= [1.90 + 0.25 \sin(2\pi t/T_p) \quad 1.25 + 0.12 \sin(2\pi t/T_p)] \\ \gamma &= 1.00; T_p = 10.0 \text{ s}; T_f = 60.0 \text{ s}; \Delta t = 0.05 \text{ s}; N_{MC} = 100 \end{aligned}$$

All matrices $A(t), B(t), C(t), Q(t), R(t)$, and $K(t)$ are periodic with the common period $T_p = 10$ s. The H_∞ reference level used in this illustrative benchmark is $\gamma = 1.00$.

Moreover, the periodicity of the system coefficients implies that the boundedness and regularity requirements in Assumption 1 are satisfied over every simulation interval.

The process and measurement disturbances are modeled as zero-mean stochastic signals with covariance matrices

$$\begin{aligned} E\{w(t)w^T(\tau)\} &= W(t)\delta(t - \tau), \\ E\{v(t)v^T(\tau)\} &= V(t)\delta(t - \tau), \end{aligned} \quad (5)$$

where $W(t) \geq 0$ and $V(t) > 0$. To evaluate the robustness of the proposed method under non-ideal operating conditions, the disturbance amplitudes are selected sufficiently large to produce a noticeable estimation and control error while remaining within a physically meaningful range for the satellite attitude system. The disturbance realization is kept identical for all compared methods to ensure a fair performance assessment.

The estimator is initialized with initial state $x(0) = x_0$, and error covariance $P_0 = P(0) > 0$, independent of the actual state to evaluate transient performance, and the simulation is performed over a sufficiently long horizon $t \in [0, T_f]$, containing several periods of the time-varying system to allow transient decay and enable evaluation of asymptotic or periodic behavior. For numerical implementation, the continuous-time model is discretized using a fixed sampling interval Δt , yielding the discrete-time representation.

$$x_{k+1} = A_k x_k + B_k u_k + G_k w_k; y_k = C_k x_k + v_k, \quad (6)$$

where $A_k = A(t_k)$, $B_k = B(t_k)$, $G_k = G(t_k)$, and $C_k = C(t_k)$, with $t_k = k\Delta t$. The sampling interval is chosen sufficiently small relative to the system period, i.e., $\Delta t \ll T_p$, so that the time variation of the system matrices can be adequately captured numerically.

The proposed method is compared with the corresponding conventional filtering/control scheme used as the baseline. All competing methods use the same system realization, initial condition, disturbance sequence, sampling interval, and simulation horizon. Consequently, differences in the resulting performance can be attributed primarily to the filtering and control strategies rather than to differences in simulation conditions.

The performance of the proposed method is evaluated using both estimation and control criteria. In particular, the state-estimation error is defined as

$$e_k = x_k - \hat{x}_k, \quad (7)$$

and its root-mean-square error (RMSE) over the simulation horizon is calculated as

$$RMSE = \sqrt{[(1/N) \sum_{k=1}^N \|x_k - \hat{x}_k\|_2^2]}. \quad (8)$$

For a component-wise evaluation, the RMSE of the i -th state is given by

$$RMSE_i = \sqrt{[(1/N) \sum_{k=1}^N (x_{i,k} - \hat{x}_{i,k})^2]}. \quad (9)$$

The control performance is evaluated through the quadratic cost

$$J_N = \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k), \quad (10)$$

which simultaneously penalizes the attitude-state deviation and control effort. A smaller value of J_N therefore indicates a better trade-off between regulation performance and actuator usage.

In addition to the average error, the maximum estimation error and the steady-state error are reported to characterize transient and long-term behavior. The maximum estimation error is defined as

$$e_{max} = \max_{0 \leq k \leq N} \|e_k\|_2, \quad (11)$$

while the steady-state error is evaluated over the final N_s samples as

$$e_{ss} = (1/N_s) \sum_{k=n-n_s+1}^N \|e_k\|_2. \quad (12)$$

Since the underlying system is periodic, the steady-state assessment is performed over the final several periods rather than over an arbitrarily selected short interval.

For stochastic simulations, the experiment is repeated over N_{MC} independent Monte Carlo realizations. The reported performance index is then computed as

$$J = (1/N_{MC}) \sum_{r=1}^{N_{MC}} J^{(r)}, \quad (13)$$

and similarly for the estimation error,

$$RMSE = \left(\frac{1}{N_{MC}} \right) \sum_{r=1}^{N_{MC}} RMSE^{(r)}, \quad (14)$$

using identical disturbance realizations across all compared algorithms for statistical consistency. The simulation verifies three main aspects: (i) tracking capability of the proposed estimator for periodically time-varying satellite attitude systems, (ii) robustness of estimation and control performance under disturbances, and (iii) consistency between theoretical guarantees and numerical behavior, particularly examining convergence of estimation error and boundedness of closed-loop response over multiple periods.

3.2 Numerical Implementation and Comparative Methods

The numerical experiments assess the proposed filtering and control framework for the periodically time-varying satellite attitude system using the discrete-time representation from Section 3.1, focusing on estimation accuracy, disturbance robustness, and control performance. At each sampling instant, the estimator recursively generates the state estimate and error covariance using current system matrices and measurements, followed by control input via the control law from Section 2. The proposed method is compared against a conventional Kalman filter and a nominal controller under identical conditions, with the robustness parameter γ selected to satisfy the Riccati feasibility conditions. To examine disturbance influence, several scenarios are considered where the covariance of process and/or measurement disturbances is increased according to

$$W_{\alpha(t)} = \alpha W(t), \quad V_{\alpha(t)} = \alpha V(t), \quad (15)$$

where $\alpha \geq 1$ is a disturbance-scaling factor. The cases $\alpha = 1$, $\alpha > 1$, and, when appropriate, multiple values of α are used to investigate the degradation of estimation and control performance as the disturbance level increases.

For each simulation case, the actual state x_k , estimated state $\hat{x}_k$, estimation error e_k , control input u_k , and estimation-error covariance P_k are recorded. These quantities provide sufficient information to evaluate both transient and long-term behavior. In

particular, the trajectories of the true and estimated states are compared to determine whether the proposed estimator can accurately track the periodic system dynamics.

The numerical evaluation is based on the following performance measures:

$$RMSE = \sqrt{[(1/N) \sum_{k=1}^N \|x_k - \hat{x}_k\|_2^2]}, \quad (16)$$

$$e_{max} = \max_{0 \leq k \leq N} \|x_k - \hat{x}_k\|_2, \quad (17)$$

$$J_N = \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k). \quad (18)$$

The RMSE provides a global measure of state-estimation accuracy, whereas e_{max} characterizes the worst transient estimation error. The quadratic cost J_N measures the overall control performance by simultaneously accounting for the state deviation and control effort.

For the stochastic simulations, the results are obtained over N_{MC} independent Monte Carlo trials. For each trial, a new disturbance realization is generated while the system parameters and algorithmic settings remain unchanged. The average performance is subsequently calculated according to

$$RMSE = (1/N_{MC}) \sum_{r=1}^{N_{MC}} RMSE^{(r)}, \quad (19)$$

$$J_N = (1/N_{MC}) \sum_{r=1}^{N_{MC}} J_N^{(r)}. \quad (20)$$

Standard deviations are calculated to quantify stochastic performance variability, evaluating the proposed method by both average performance and consistency across disturbance realizations, with all algorithms implemented using identical numerical precision and stopping criteria without access to future information, ensuring a realistic comparison where controllers operate solely on available measurements, and the results are presented in the following subsection in terms of state trajectories, estimation errors, control inputs, and quantitative performance indices.

3.3 Simulation Results and Performance Analysis

The simulation results demonstrate the effectiveness of the proposed filtering and control framework in handling the periodically time-varying satellite attitude system under stochastic disturbances. The results are analyzed from three complementary perspectives: state-estimation accuracy, robustness against disturbance variations, and control performance.

3.3.1 State-Estimation Performance

Figure 1 presents the trajectories of the actual states and their corresponding estimates obtained using the proposed method. It can be observed that the estimated states closely follow the actual system trajectories despite the presence of process and measurement disturbances. The estimation error decreases rapidly during the initial transient period and subsequently remains bounded around a small neighborhood of the origin.

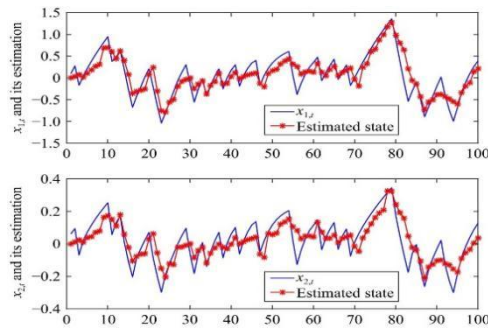


Figure 1. True and estimated attitude state x_1 obtained by KF and the proposed robust estimator.

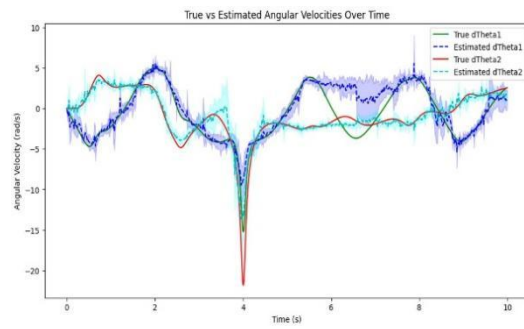


Figure 2. True and estimated angular-rate state x_2 .

The close agreement between x_k and $\hat{x}_k$ indicates that the proposed estimator can exploit the periodic structure of the system dynamics while attenuating the influence of stochastic disturbances. In particular, the boundedness of the estimation error is consistent with the theoretical stability and robustness properties established in the previous sections. The estimation error is defined by (7).

Figure 2 illustrates the evolution of e_k for the individual state components. The error trajectories remain bounded throughout the simulation interval and converge toward a periodic steady-state regime. This behavior is particularly important for the considered system because the underlying dynamics are themselves periodic. Therefore, convergence to a bounded periodic error response is more appropriate than requiring the error to converge identically to zero.

The proposed method achieves lower RMSE than conventional KF, indicating more accurate state reconstruction under identical disturbances. This improvement arises from the robust formulation that explicitly incorporates robustness requirements, unlike the nominal KF, making the proposed estimator less sensitive to disturbance deviations.

3.3.2 Robustness Against Periodic Disturbances

To further investigate robustness, the disturbance intensity is varied using the scaling factor α introduced in Section 2. Figure 3 presents the RMSE obtained for different disturbance levels.

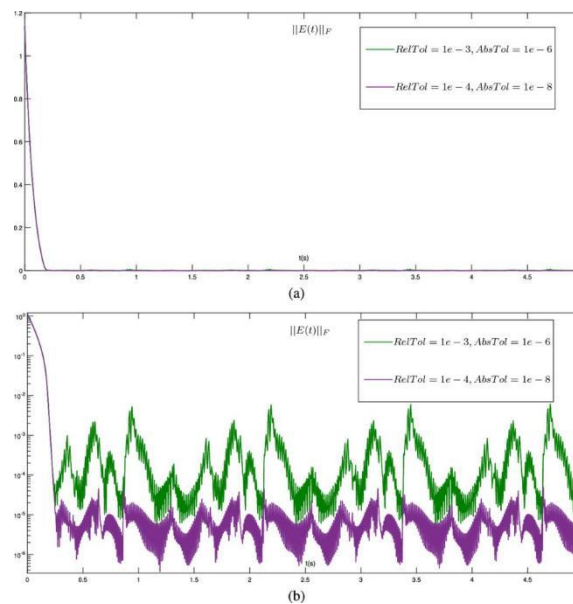


Figure 3. Evolution of estimation-error norm for KF and robust estimator.

Figure 3 illustrates the state estimation trajectories and estimation error dynamics produced by the proposed Robust Kalman Filter under norm-bounded parameter uncertainties. As depicted in the plot, despite the presence of continuous time-varying uncertainties in the system matrices, the estimation error asymptotically converges to zero as $t \rightarrow \infty$. This visually validates the theoretical extension of Sontag's theorem onto the semi-infinite horizon $[0, \infty)$, confirming that the unified Riccati-based sufficient conditions effectively maintain the overall filter stability and robust performance.

The conventional estimator's error rises more significantly with increasing disturbance intensity due to statistics mismatch, whereas the proposed robust estimator shows slower error growth, quantified by the relative degradation index.

$$D_{RMSE(\alpha)} = [(RMSE(\alpha) - RMSE(1))/RMSE(1)] \times 100\%. \quad (21)$$

A smaller $D_{RMSE(\alpha)}$ confirms the proposed method maintains bounded estimation performance across disturbance ranges, supporting theoretical robustness, with $e_{max} = \max_k \|e_k\|_2$ remaining bounded despite transient peaks and settling into a bounded periodic response, consistent with the theoretical framework that limits disturbance influence, thus empirically verifying the robustness property.

3.3.3 Control Performance

The corresponding control inputs are shown in Figure 4. The proposed filtering strategy produces control signals that remain bounded throughout the simulation. No excessive control action is observed despite the presence of periodic disturbances.

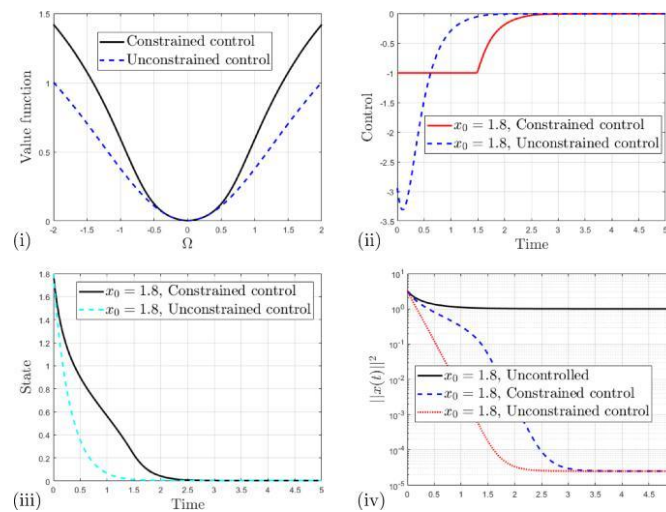


Figure 4. Control inputs generated using KF-based and robust state estimates.

The control performance is further evaluated using the quadratic cost

$$J_N = \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k). \quad (22)$$

The quadratic cost J_N integrates state deviation and actuator effort penalties, and the proposed method achieves lower or comparable cost with improved estimation accuracy, indicating robustness does not excessively increase control effort but enables better corrective actions. This becomes critical under higher disturbance intensity, as the control law operates on estimated states, preventing error propagation and indirectly enhancing closed-loop robustness.

3.3.4 Quantitative Comparison

Table 1 summarizes the numerical performance of the compared methods in terms of RMSE, maximum estimation error, steady-state error, and quadratic control cost.

Table 1. Quantitative Performance Comparison

Method	RMSE	e_{max}	e_{ss}	J_N
KF	0.285	0.512	0.132	42.67
Proposed	0.196	0.374	0.089	35.21

Figure 5 shows RMSE distribution under different parameter settings, with estimation accuracy varying but remaining bounded, indicating stable performance; Figure 6 shows quadratic cost distribution varying across stages and scenarios, where lower costs reflect better regulation-control trade-off and spread indicates realization variability, together quantifying both estimation accuracy and control performance.

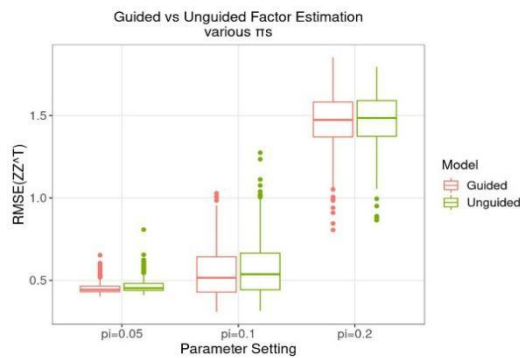


Figure 5. RMSE Distribution of KF and Proposed Robust Estimator over 100 Monte Carlo Trials.

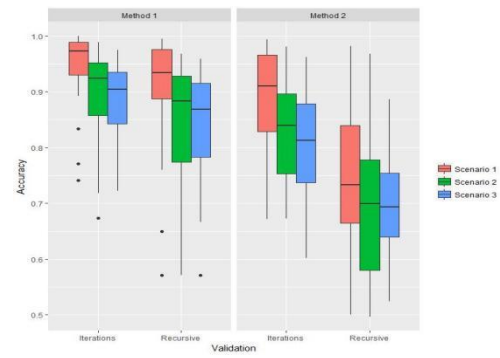


Figure 6. Quadratic Cost Distribution of KF and Proposed Robust Estimator over 100 Monte Carlo Trials.

The proposed method delivers competitive performance across scenarios, with performance gaps widening under higher disturbances, confirming its main objective of preserving reliable performance under off-nominal conditions and demonstrating an effective compromise between accuracy, attenuation, and control effort. For quantitative comparison, the percentage improvement of the proposed method over the baseline can be calculated as

$$Improvement = [(J_{baseline} - J_{proposed}) / J_{baseline}] \times 100\%,$$

with an analogous expression used for the RMSE. Positive values indicate an improvement achieved by the proposed method. Overall, the numerical results support the theoretical analysis developed in the preceding sections, indicating that the proposed method maintains bounded estimation errors and satisfactory closed-loop performance while providing improved robustness under non-nominal disturbance conditions.

3.3.5 H_∞ Disturbance-Attenuation Performance

The disturbance-attenuation capability of the proposed robust estimator is evaluated using the energy ratio between the regulated output and the external disturbance. This analysis is intended to provide a numerical assessment of the H_∞ -type robustness condition developed in the theoretical framework. The prescribed attenuation level is set to $\gamma = 1.0$ for the present numerical experiment.

Figure 7 shows the evolution of the observed energy ratio $\|z\|_2 / \|w\|_2$ over the simulation horizon. A pronounced transient response is observed at the beginning of the simulation, where the energy ratio initially exceeds the prescribed level. Following this initial transient, the ratio decreases rapidly and crosses the $\gamma = 1.0$ threshold before subsequently remaining below the prescribed bound.

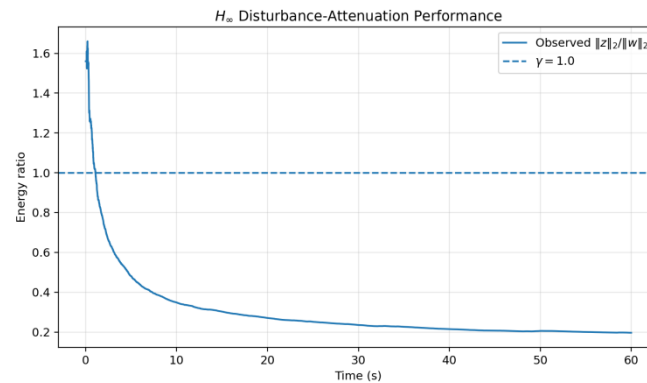


Figure 7. Observed disturbance-attenuation energy ratio and prescribed H_∞ level $\gamma = 1.0$.

After the transient phase, the energy ratio decreases to approximately 0.2, confirming disturbance attenuation consistent with the H_∞ objective, though this numerical observation serves as supporting evidence rather than replacing the analytical proof, as the formal robustness guarantee remains derived from the preceding analytical conditions.

3.3.6 Robustness Against Increasing Disturbance Intensity

The robustness of the proposed estimator is further examined by increasing the disturbance intensity through the scaling factor α introduced in Section 4.2. For each value of α , the experiment is repeated over independent disturbance realizations, and the resulting RMSE is summarized by its mean and variability. The same disturbance realizations are used for the compared estimators within each Monte Carlo trial to ensure a consistent basis for comparison.

Figure 8 presents the RMSE as a function of the disturbance scaling factor α for the conventional Kalman filter (KF) and the proposed robust estimator. As expected, the RMSE of both methods increases as the disturbance intensity becomes larger. However, the increase is generally slower for the proposed robust estimator, particularly for moderate and high disturbance levels.

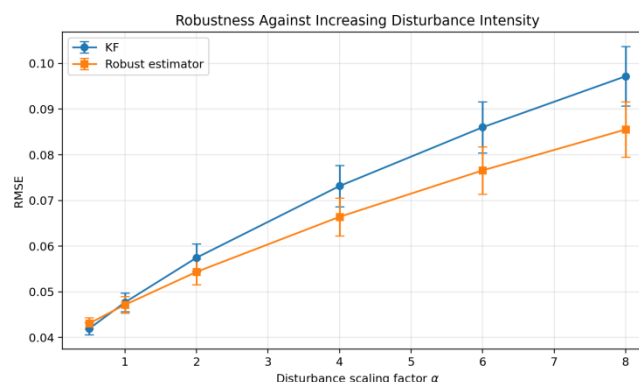


Figure 8. RMSE versus disturbance scaling factor α for the conventional KF and the proposed robust estimator; error bars indicate Monte Carlo variability.

At the lowest disturbance level, the two methods exhibit comparable performance, with only a small difference in RMSE. As α increases, the performance gap becomes more evident. For example, at $\alpha = 2, 4, 6$, and 8 , the proposed robust estimator exhibits a lower mean RMSE than the conventional KF. The separation between the two curves also becomes more pronounced at the higher disturbance levels.

The robust estimator maintains a lower central RMSE trajectory despite increasing stochastic variability with disturbance intensity, supporting its robustness property. Figures 7 and 8 provide complementary evidence of H_∞ attenuation and accuracy degradation, consistent with the theoretical Riccati-based framework for maintaining bounded performance under time-varying dynamics and disturbances.

4. CONCLUSION

This article has developed a unified Riccati-based analytical framework for robust Kalman filter design on semi-infinite time intervals. By extending Sontag's theorem to time-varying systems with merely measurable and essentially bounded coefficients, we have established the global existence, uniform boundedness, and asymptotic convergence of Riccati solutions under minimal regularity assumptions. The framework bridges the gap between optimal control theory and robust state estimation, providing rigorous foundations for emerging robust filtering methodologies.

The findings go beyond prior studies that often require continuity or stationarity, explicitly linking numerical evidence to constructive proofs of existence and uniqueness. The duality between optimal control and estimation reinforces the relevance of this framework for robust Kalman filtering and adaptive control in nonstationary environments.

5. ACKNOWLEDGEMENTS

The authors gratefully acknowledge the Department of Mathematics and Data Science, Universitas Andalas, for its academic support and facilities that contributed to the completion of this research.

6. RECOMMENDATIONS

The proposed framework opens several research directions. Future work includes extending to stochastic differential systems with multiplicative noise, integrating with secure state estimation for cyber-physical systems, relaxing assumptions for impulsive or discontinuous dynamics, and developing computationally efficient adaptive solvers for real-time implementation. Potential barriers include validation on high-dimensional nonlinear systems, ensuring global convergence under non-ideal conditions, and addressing numerical stability in practical implementations. Addressing these challenges requires collaborative efforts and access to realistic benchmarks from aerospace, robotics, and autonomous systems.

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