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The Curve Estimation Nonparametric Regression Multiresponse Mixed with Truncated Spline, Fourier Series, and Kernel

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Abstract
This study formulates a nonparametric regression model for multiresponse data by combining three estimators: truncated spline, Fourier series, and kernel function. Each estimator captures specific characteristics. Truncated spline capture local traits with knot points, while fourier series capture periodic patterns and kernel estimators provide flexible smoothing for unknown functional forms. The model proposed is under an additive assumption where each predictor contributes independently to each response. Estimation is done with Weighted Least Squares (WLS) method which is efficient in managing the correlations between the multiresponse variables. The final multiresponse nonparametric regression curve estimator combining truncated spline, Fourier series, and kernel is given by $\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{f}} + \hat{\boldsymbol{g}} + \hat{\boldsymbol{h}}$ obtained by solving the WLS optimization problem: $\min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\{\boldsymbol{\epsilon}'\boldsymbol{W}\boldsymbol{\epsilon}\} = \min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\{(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})'\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})\}$. The solution to this problem results in the mixed estimator, which can be expressed as: $\hat{\boldsymbol{\mu}} = \boldsymbol{Ey}$ with $\boldsymbol{E} = \boldsymbol{UB} + \boldsymbol{ZA} + \boldsymbol{T}$.

Keywords: Fourier Series; Kernel Smoothing; Multiresponse Nonparametric Regression; Truncated Spline; Weighted Least Square.

Abstrak
Studi ini merumuskan model regresi nonparametrik untuk data multirespon dengan menggabungkan tiga estimator: spline truncated, deret Fourier, dan fungsi kernel. Setiap estimator digunakan untuk menangkap karakteristik tertentu: spline truncated menangkap pola lokal melalui titik knot, deret Fourier menangkap pola periodik, sedangkan estimator kernel memberikan pelandaian yang fleksibel untuk bentuk fungsi yang tidak diketahui. Model yang diusulkan didasarkan pada asumsi aditif, di mana setiap peubah prediktor memberikan kontribusi secara independen terhadap masing-masing respon. Estimasi dilakukan dengan metode Weighted Least Squares (WLS) yang efisien dalam mengelola korelasi antar variabel respon. Estimator kurva regresi nonparametrik multirespon akhir yang menggabungkan spline truncated, deret Fourier, dan kernel diberikan oleh: $\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{f}} + \hat{\boldsymbol{g}} + \hat{\boldsymbol{h}}$ yang diperoleh melalui penyelesaian masalah optimisasi WLS berikut: $\min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\{\boldsymbol{\epsilon}'\boldsymbol{W}\boldsymbol{\epsilon}\} = \min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\{(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})'\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})\}$. Solusi dari permasalahan ini menghasilkan estimator campuran, yang dapat dinyatakan sebagai: $\hat{\boldsymbol{\mu}} = \boldsymbol{Ey}$ dengan $\boldsymbol{E} = \boldsymbol{UB} + \boldsymbol{ZA} + \boldsymbol{T}$.

Kata Kunci: Deret Fourier; Kernel; Kuadrat Terkecil dengan Bobot; Regresi Nonparametrik Multirespon; Spline Truncated

1. INTRODUCTION

Statistics includes a various of disciplines such as time series, stochastic processes, probability, experimental design, reliability, and regression analysis (Budiantara, 2009). Of these, regression analysis occupies an important place as it is concerned with the relationship between one or more predictor variables and a response variable. Regression models are typically classified as parametric, semiparametric, or nonparametric. When the model's functional form such as linear, quadratic, or polynomial is known, parametric regression is appropriate. When the underlying relationship between variables is unknown, however, nonparametric regression is the preferred method (Eubank, 1999).

Since nonparametric regression does not force a predetermined structure on the data, it provides more flexibility in curve estimation. Nonparametric models minimize subjective bias by letting the data dictate the curve's shape, as (Eubank, 1999) pointed out. Numerous methods, including spline functions, Fourier series, kernel estimators, and others, have been developed over time for estimating nonparametric regression curves (Budiantara, 2009; Green & Silverman, 1993; Härdle, 1990; Wahba, 1990) .

The truncated spline approach is a popular estimator that is renowned for its capacity to identify localized patterns in segmented data (I. N. Budiantara, 2004). The effectiveness of this technique depends on the number and placement of knots (Fitriyani & Budiantara, 2014; Montoya et al., 2014). In contrast, the Fourier series estimator works well for simulating oscillatory behavior and periodic patterns in data (Asrini & Budiantara, 2014; Bilodeau, 1992). Although it is less flexible for non-periodic trends, it offers strong interpretability when repeated data behavior is seen. In the meantime, the kernel estimator is a potent smoothing method that works especially well when there is no discernible pattern in the data and can be readily adjusted to unknown functional forms (Budiantara & Mulianah, 2007; Härdle, 1990).

Despite the advantages of truncated spline, Fourier series, and kernel estimators, existing studies have primarily focused on single-response or bi-response models, neglecting the complexities of multiresponse data where correlations between responses are significant. This gap limits the applicability of nonparametric regression in real-world scenarios where multiple interrelated outcomes are common. The study addresses this limitation by integrating these estimators within a multiresponse framework, leveraging the Weighted Least Squares (WLS) method to account for inter-response correlations. This approach not only enhances flexibility but also improves estimation efficiency, offering a more comprehensive solution for multivariate data analysis.

Every one of these estimators has advantages and disadvantages. Kernel methods are sensitive to bandwidth selection, Fourier series may have trouble with irregular patterns, and truncated splines may cause oscillations close to data boundaries. This encourages the creation of a mixed estimator approach, which combines the three techniques to improve overall model robustness and make up for their respective shortcomings.

To better handle complex data structures, recent research has suggested combining two or more estimators in nonparametric regression (Adrianingsih et al., 2021; Budiantara et al., 2015; Nurcahayani et al., 2021). Nevertheless, single-response or bi-response models have been the focus of most of these efforts. The benefit of applying this method to a multiresponse setting is that it preserves the flexibility of nonparametric estimation while capturing joint variation among correlated responses.

This study uses the Weighted Least Squares (WLS) method to estimate the parameters in such a situation. By using a variance–covariance weighting structure, WLS enables us to take into consideration response specific variances as well as correlations across multiple responses. To increase estimation efficiency and coherence, the process first addresses individual component estimation before fine-tuning the model under a weighted system.

While prior studies (Budiantara et al., 2015; Nurcahayani et al., 2021) demonstrated the utility of mixed estimators for single-response data, their approaches fail to account for correlated multiresponse structures. This model bridges this gap by integrating truncated splines, Fourier series, and kernel smoothing under a WLS framework, enabling joint estimation while preserving flexibility. Developing a mixed estimator model with truncated spline, Fourier series, and kernel components for multiresponse nonparametric regression, this model provides a thorough framework for capturing complex data behavior because it is made to adjust to different functional patterns in the predictor-response relationship.

2. MATERIAL AND METHODS

2.1 Truncated Spline Function

Since a truncated spline is a continuous piecewise polynomial segment, splines can handle data patterns that exhibit increases or decreases with the aid of knot points, and the curve that results is comparatively smooth. Knot points are typical intersections that show shifts in the function's pattern of behavior over various time periods. The following form is a general way to express the truncated spline regression equation of order p_j for the function f (Härdle, 1990; Wahba, 1990).

$$f_j(u_{ai}) = \sum_{v=0}^{p_j} \beta_{vaj} u_{ai}^{v} + \sum_{l=1}^{s} \lambda_{alj} \, (u_{ai} - K_{alj})_{+}^{p_j} \tag{1}$$

The knot points are denoted by K, which are the points where the pattern of the function changes in the spline model. The number of knots used in this model is denoted as s, while

p_j represents the number of spline orders used for the response variable j, with $j = 1,2, \ldots m$. Each data point in the analysis has an index i, which indicates the order of the data from $i = 1,2,3, \ldots n$, with n being the total number of data points used. The response variables in the model amount to m, which reflect the number of aspects or characteristics being estimated. The parameters in this model are denoted by β and λ, which play a role in determining the shape and flexibility of the regression curve estimation. with the function $(u_{ai} - K_{alj})_+^{p_j}$ being a truncated function given by:

$$(u_{ai} - K_{alj})_+^{p_j} = \begin{cases} (u_{ai} - K_{alj})_+^{p_j} & , u_{ai} \geq K_{alj} \\ 0 & , u_{ai} < K_{alj} \end{cases} \quad (2)$$

2.2 Fourier Series Function

The Fourier series is a trigonometric polynomial function that has a high degree of flexibility. Fourier series are usually used for unknown data patterns and tend to have seasonal patterns in the observed data. With the expansion into the form of a Fourier series, a periodic function can be expressed as the sum of several harmonic functions, namely sine and cosine functions (Bilodeau, 1992). According to (Tripena & Budiantara, 2006), a multivariable nonparametric regression model is given:

$$\begin{aligned} y_i &= g(z_{1i}, z_{2i}, \ldots, z_{qi}) + \varepsilon_i \\ &= \sum_{b=1}^{q} g(z_{bi}) + \varepsilon_i, b = 1,2, \ldots, q \end{aligned} \quad (3)$$

A Fourier series function approximates the regression curve $g(z_{ji})$, which is assumed to be unknown and to belong to the space of continuous functions $C(0, \pi)$, with random error ε_ii assumed to be independently normally distributed with mean 0 and variance σ^2.

$$g_j(z_{bi}) = \gamma_{bj} z_{bi} + \frac{1}{2} \delta_{0bj} + \sum_{\theta=1}^{e} \delta_{\theta bj} \cos \theta z_{li}, \quad (4)$$

with $\gamma_j, \delta_{0j}, \delta_{\theta j}$ being the model parameters.

2.3 Kernel Function

In statistics, kernel regression is a nonparametric method for calculating a random variable's conditional expectation. Finding a nonlinear relationship between two random variables, x and y, is the aim in order to determine and apply the proper weights. The kernel K is generally defined as follows (Bowman & Azzalini, 1997):

$$K_\psi(t) = \frac{1}{\psi} K \left(\frac{t}{\psi}\right); \ -\infty < t < \infty, \psi > 0$$

Kernels can perform a variety of functions, including triweight, cosine, quadratic, Gaussian, Epanechnikov, and uniform kernels (Härdle, 1990). Equation (5) defines the Gaussian Kernel function, which is superior to models that use other kernel functions (Hidayat et al, 2020; Wand and Jones, 1994).

$$K(t) = \frac{1}{\sqrt{2\pi}} exp\left(\frac{1}{2}(-t^2)\right); -\infty < t < \infty \tag{5}$$

Only local constants are present in the Nadaraya-Watson Kernel regression model, which is a local polynomial model. Consequently, the function can be minimized using Equation (6) if $h(t_i)$ only contains local constants:

$$h(t_i) = \sum_{o=1}^{n} n^{-1} \left[\frac{K_\psi(t - t_o)}{n^{-1}\sum_{m=1}^{n} K_\psi(t - t_m)}\right] y_i = n^{-1} \sum_{i=o}^{n} V_{\psi o}(t) y_o \tag{6}$$

2.4 Methods

- Given paired data $(u_{1i}, \dots, u_{pi}, z_{1i}, \dots, z_{qi}, t_{1i}, \dots t_{ri}, y_{1i}, \dots, y_{mi})$ and the relationship between predictor variables $(u_{1i}, \dots, u_{pi}, z_{1i}, \dots, z_{qi}, t_{1i}, \dots t_{ri}, y_{1i}, \dots, y_{mi})$ and response variables $(y_{1i}, \dots, y_{mi})$ meets the requirements of the multiresponse nonparametric regression model, as done by Equation (7).

$$y_{ji} = \sum_{a=1}^{p} f_j(u_{ai}) + \sum_{b=1}^{q} g_j(z_{bi}) + \sum_{c=1}^{r} h_j(t_{ci}) + \epsilon_{ji} \tag{7}$$

- Component $\sum_{a=1}^{p} f_j(u_{ai})$ is approximated with a truncated spline function, the component $\sum_{b=1}^{q} g_j(z_{bi})$ is approximated with a Fourier series function, and the component ∑_(c=1) ^ $\sum_{c=1}^{r} h_j(t_{ci})$ is approximated with a kernel function.

- Arranging a mixed multiresponse nonparametric regression model into vector form as follows:

$$\boldsymbol{y} = \boldsymbol{f} + \boldsymbol{g} + \boldsymbol{h} + \boldsymbol{\varepsilon}$$

  where $\boldsymbol{y} = [\boldsymbol{y}_{11}\ \boldsymbol{y}_{12} \dots \boldsymbol{y}_{1n} \dots\ \boldsymbol{y}_{m1}\ \boldsymbol{y}_{m2} \dots \boldsymbol{y}_{mn}]^T$ and $\boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_{11}\ \boldsymbol{\varepsilon}_{12} \dots \boldsymbol{\varepsilon}_{1n} \dots\ \boldsymbol{\varepsilon}_{m1}\ \boldsymbol{\varepsilon}_{m2} \dots \boldsymbol{\varepsilon}_{mn}]^T$.

- Put the mixed multiresponse nonparametric regression model in the form of $\boldsymbol{y}^* = \boldsymbol{U\beta} + \boldsymbol{Z\theta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{y}^* = (\boldsymbol{I} - \boldsymbol{T})\boldsymbol{y}$.

- Forming the variance-covariance matrix $\boldsymbol{W}$ as a weighting.

$$\boldsymbol{W} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1m} & \sigma_{2m} & \cdots & \sigma_{mm} \end{pmatrix} \otimes \boldsymbol{I}$$

- Solve the following optimization to estimate $\widehat{\boldsymbol{\beta}}$ using the WLS method.

$$\underset{\boldsymbol{\beta}}{\mathbf{Min}}\{(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})^{\boldsymbol{T}}\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})\}$$

- Completing the calculation of the partial first derivative and set to zero.

$$\frac{\partial\left((\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})^{\boldsymbol{T}}\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})\right)}{\partial\boldsymbol{\beta}}$$

- Solve the following optimization to estimate $\widehat{\boldsymbol{\theta}}$ using the WLS method.

$$\underset{\boldsymbol{\theta}}{\mathbf{Min}}\{(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})^{\boldsymbol{T}}\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})\}$$

- Completing the calculation of the partial first derivative and set to zero.

$$\frac{\partial\left((\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})^{\boldsymbol{T}}\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\theta})\right)}{\partial\boldsymbol{\theta}}$$

- Obtained estimators in nonparametric multivariate response regression with truncated spline, Fourier series, and kernel:

$$\widehat{\boldsymbol{y}}_{ji} = \sum_{a=1}^{p} \widehat{\boldsymbol{f}}_j(\boldsymbol{u}_{ai}) + \sum_{b=1}^{q} \widehat{\boldsymbol{g}}_j(\boldsymbol{z}_{bi}) + \sum_{c=1}^{r} \widehat{\boldsymbol{h}}_j(\boldsymbol{t}_{ci})$$

2.4 Data Source

This study uses secondary data in 2024 from publications by the Statistics of Indonesia (BPS). The model obtained is applied to Poverty Indicators in 38 districts/cities in East Java. This study uses the Percentage of Poor People (Y_1), Poverty Depth Index (Y_2), and Poverty Severity Index (Y_3) as composite response variables to measure Poverty Indicators in East Java. The predictor variables used are the Life Expectancy (X_1), the Mean of Years Schooling (X_2), and the Labor Force Participation Rate (X_3).

There are several combinations for the model to find which variables match the estimator used. There is also a combination of knot points and oscillations that are limited to using only 1 knot point with 1 oscillation and 2 knot points with 2 oscillations. So that it produces the model in the following Table 1.

Table 1. Every Combination of Trial Models

Model	Variables			Number of Models
	Truncated Spline	Fourier Series	Kernel	
A.1	X_1	X_2	X_3	2
A.2	X_1	X_3	X_2	2
A.3	X_2	X_1	X_3	2
A.4	X_2	X_3	X_1	2
A.5	X_3	X_2	X_1	2
A.6	X_3	X_1	X_2	2
B	X_1, X_2, X_3			2
C		X_1, X_2, X_3		2
D			X_1, X_2, X_3	1
	Total of Model			17

2.5 Smoothing Parameter Selection

An essential step in nonparametric regression modeling involves determining the optimal configuration for knot placement, oscillation level, and smoothing parameters. A large smoothing parameter results in an overly smooth estimator that may underfit the data, whereas a small smoothing parameter produces a rough estimator that is more sensitive to noise and may lead to overfitting (Wahba, 1990). Therefore, selecting an appropriate method for tuning the smoothing parameters is crucial. In this study, the smoothing parameter selection is performed using the Generalized Cross Validation (GCV) approach, as previously implemented by (Wahba, 1990), with its optimality properties comprehensively discussed by (Craven & Wahba, 1978). The GCV method is adapted here to accommodate the structure of the proposed model, which integrates three types of estimators truncated spline, Fourier series, and kernel functions within a unified multiresponse nonparametric regression framework. The modified GCV formulation used to optimize this combined estimator is presented in the following section.

$$GCV(\boldsymbol{K}, \boldsymbol{\theta}, \boldsymbol{\psi}) = \frac{MSE(\boldsymbol{K}, \boldsymbol{\theta}, \boldsymbol{\psi})}{(N^{-1} tr[\boldsymbol{I} - \boldsymbol{E}(\boldsymbol{K}, \boldsymbol{\theta}, \boldsymbol{\psi})])^2} \quad (8)$$

where

$$MSE(\boldsymbol{K}, \boldsymbol{\theta}, \boldsymbol{\psi}) = N^{-1} \boldsymbol{y}^T \left(\boldsymbol{I} - \boldsymbol{E}(\boldsymbol{K}, \boldsymbol{\theta}, \boldsymbol{\psi})\right)^T \left(\boldsymbol{I} - \boldsymbol{E}(\boldsymbol{K}, \boldsymbol{\theta}, \boldsymbol{\psi})\right) \boldsymbol{y}$$

For the mixed estimator, GCV optimizes knot locations (spline), oscillation count (Fourier), and bandwidth (kernel) jointly by minimizing Equation (8), where the smoothing matrix $\boldsymbol{E}$ accounts for all three components.

3. RESULT

3.1 Model of Combined Truncated Spline, Fourier Series, and Kernel Estimators

The regression curve estimators defined in Equations (1), (4), and (6), which represent the truncated spline, Fourier series, and kernel components respectively, depend on their associated parameters, such as the spline coefficients, Fourier amplitudes, and the smoothing matrix. Consequently, the estimator of the combined regression model in Equation (7) is obtained by applying the Weighted Least Squares (WLS) method. For this purpose, the following lemmas and theorems are established to derive the closed-form solutions of the estimated regression curve.

Given paired data $(u_{1i},\dots,u_{pi},z_{1i},\dots,z_{qi},t_{1i},\dots t_{ri},y_{1i},\dots y_{mi})$, where each data component consists of three predictor groups and one response group. First, the vector $\boldsymbol{u}_i = [u_{1i},u_{2i},\dots,u_{pi}]$ predictor variables that form the truncated spline components. Then, the vector $\boldsymbol{z}_i = [z_{1i},z_{2i},\dots,z_{qi}]$ represents the predictor variables used as kernel components. Next, the vector $\boldsymbol{t}_i = [t_{1i},t_{2i},\dots t_{ri}]$ predictor variables that are components of the Fourier series. Finally, the vector $\boldsymbol{y}_i = [y_{1i},y_{2i},\dots,y_{mi}]$ represents the multivariate response to be modeled. The multiresponse nonparametric regression model for the data is shown in Equation (9).

Let y_{ji} denote the j-th response variable for the i-th observation. Let u_{aj}, z_{bj}, and t_{cj} denote the predictor variables associated respectively with truncated spline, Fourier series, and kernel estimators. The general additive model is:

$$y_{ji} = \boldsymbol{\mu}_j(\boldsymbol{u}_i,\boldsymbol{z}_i,\boldsymbol{t}_i) + \epsilon_{ji}, \epsilon_{ji} \sim N(0,\sigma_j^2) \tag{9}$$

The random error component ϵ_{ji} detailed in (Wang et al., 2000), in Equation (9) assuming $\epsilon_{ji} \sim N(0,\sigma_j^2)$. When considering multiple responses y_{ji} and $y_{\tau i}$, the errors may be correlated. The correlation between the j-th and τ-th response errors for the same observation i is defined as $corr(\varepsilon_{ji}\varepsilon_{\tau i}) = \rho$, for $j \neq \tau$, and zero otherwise with $j,\tau = 1,2,\dots,m$. This structure implies a consistent inter-response correlation across observations. The correlation coefficient ρ is defined by $\rho = \frac{cov(\varepsilon_{ji}\varepsilon_{\tau i})}{\sqrt{\sigma_j^2\sigma_\tau^2}}$, leading to a covariance term $\sigma_{j\tau} = \rho\sqrt{\sigma_j^2\sigma_\tau^2}$. Where $\mu_{ij}(u_{pi},z_{qi},t_{ri})$ is an unknown regression function assumed to be additive:

$$\begin{aligned}\boldsymbol{\mu}_j(\boldsymbol{u}_i,\boldsymbol{z}_i,\boldsymbol{t}_i) &= f_j(u_{1i},\ldots,u_{pi}) + g_j(z_{1i},\ldots,z_{qi}) + h_j(t_{1i},\ldots t_{ri})\\ &= f_j(u_{1i}) + \cdots + f_j(u_{pi}) + g_j(z_{1i}) + \cdots + g_j(z_{qi}) + h_j(t_{1i}) + \cdots + h_j(t_{ri})\\ &= \sum_{a=1}^{p} f_j(u_{ai}) + \sum_{b=1}^{q} g_j(z_{bi}) + \sum_{c=1}^{r} h_j(t_{ci})\end{aligned} \tag{10}$$

As stated in Equation (10), the multiresponse nonparametric regression function consists of the sum of truncated spline components, Fourier series, and kernels. In the component $f_j(u_{ai})$, it will be approximated with a linear truncated spline function with s knot points presented in Equation (11).

$$f_j(u_{ai}) = \beta_{0aj} + \beta_{1aj}u_{ai} + \sum_{l=1}^{s} \lambda_{laj}\left(u_{ai} - K_{laj}\right)_+ \tag{11}$$

The function $f_j(u_{ai})$ with one predictor variable, symbolized as a, can be written in the following matrix form:

$$\boldsymbol{f}_{aj}(\boldsymbol{u}_a) = \begin{bmatrix} f_{aj}(u_{a1}) \\ f_{aj}(u_{a2}) \\ \vdots \\ f_{aj}(u_{an}) \end{bmatrix} = \begin{bmatrix} \beta_{0aj} + \beta_{1aj}u_{a1} + \sum_{l=1}^{s} \lambda_{laj}\left(u_{a1} - D_{laj}\right)_+ \\ \beta_{0aj} + \beta_{1aj}u_{a2} + \sum_{l=1}^{s} \lambda_{laj}\left(u_{a2} - D_{laj}\right)_+ \\ \vdots \\ \beta_{0aj} + \beta_{1aj}u_{an} + \sum_{l=1}^{s} \lambda_{laj}\left(u_{an} - D_{laj}\right)_+ \end{bmatrix}$$

$$= \begin{bmatrix} 1 & u_{a1} & \left(u_{a1} - D_{1aj}\right)_+ & \left(u_{a1} - D_{2aj}\right)_+ & \cdots & \left(u_{a1} - D_{saj}\right)_+ \\ 1 & u_{a2} & \left(u_{a2} - D_{1aj}\right)_+ & \left(u_{a2} - D_{2aj}\right)_+ & \cdots & \left(u_{a2} - D_{saj}\right)_+ \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_{an} & \left(u_{an} - D_{1aj}\right)_+ & \left(u_{an} - D_{2aj}\right)_+ & \cdots & \left(u_{an} - D_{saj}\right)_+ \end{bmatrix} \begin{bmatrix} \beta_{0aj} \\ \beta_{1aj} \\ \lambda_{1aj} \\ \lambda_{2aj} \\ \vdots \\ \lambda_{saj} \end{bmatrix}$$

$$= \boldsymbol{U}_{aj}\boldsymbol{\beta}_{aj} \tag{12}$$

Likewise, the truncated spline function for the p number of predictors nonparametric regression shown in Equation (13) for $a = 1, 2, \ldots, p$ predictors can be expressed as:

$$\begin{aligned}\boldsymbol{f}_j &= \sum_{a=1}^{p} \boldsymbol{f}_{aj} \\ &= \boldsymbol{f}_{1j} + \boldsymbol{f}_{2j} + \cdots + \boldsymbol{f}_{pj} \\ &= \boldsymbol{U}_{1j}\boldsymbol{\beta}_{1j} + \boldsymbol{U}_{2j}\boldsymbol{\beta}_{2j} + \cdots + \boldsymbol{U}_{pj}\boldsymbol{\beta}_{pj} \\ &= \sum_{a=1}^{p} \boldsymbol{U}_{aj}\boldsymbol{\beta}_{aj} \\ &= \boldsymbol{U}_j\boldsymbol{\beta}_j\end{aligned} \tag{13}$$

where $\boldsymbol{\beta}_j = \begin{bmatrix}\beta_{01j} & \beta_{11j} & \lambda_{11j} & \lambda_{21j} & \cdots & \lambda_{saj} & \cdots & \beta_{0pj} & \beta_{1pj} & \lambda_{1pj} & \cdots & \lambda_{spj}\end{bmatrix}'$ and

$$\boldsymbol{U}_j = \begin{bmatrix} 1 & u_{a1} & (u_{a1}-D_{1aj})_+ & \cdots & (u_{a1}-D_{saj})_+ & \cdots & 1 & u_{p1} & (u_{p1}-D_{1pj})_+ & \cdots & (u_{p1}-D_{spj})_+ \\ 1 & u_{a2} & (u_{a2}-D_{1aj})_+ & \cdots & (u_{a2}-D_{saj})_+ & \cdots & 1 & u_{p2} & (u_{p2}-D_{1pj})_+ & \cdots & (u_{p2}-D_{spj})_+ \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_{an} & (u_{an}-D_{1aj})_+ & \cdots & (u_{an}-D_{saj})_+ & \cdots & 1 & u_{pn} & (u_{pn}-D_{1pj})_+ & \cdots & (u_{pn}-D_{spj})_+ \end{bmatrix}$$

Equation (14) is the result of expressing the truncated multiresponse spline function in matrix form using Equation (13).

$$\begin{aligned}\boldsymbol{f} &= \begin{bmatrix}\boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_m\end{bmatrix} = \begin{bmatrix}\boldsymbol{U}_1\boldsymbol{\beta}_1 \\ \boldsymbol{U}_2\boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{U}_m\boldsymbol{\beta}_m\end{bmatrix} \\ &= \begin{bmatrix}\boldsymbol{U}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{U}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{U}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_m\end{bmatrix} \\ &= \boldsymbol{U}_{nm\times(p(s+2))m}\boldsymbol{\beta}_{(p(s+2))m\times 1}\end{aligned} \tag{14}$$

Next, the component $g_j(z_{bi})$ is approximated with a Fourier series with oscillation e in Equation (15).

$$g_j(z_{bi}) = \alpha_{bj}z_{bi} + \frac{1}{2}\delta_{0bj} + \sum_{\theta=1}^{e} \delta_{\theta bj} \cos \theta z_{bi} \tag{15}$$

If function $\boldsymbol{g}_j(\boldsymbol{z}_b) = \begin{bmatrix}g_j(z_{b1}) & g_j(z_{b2}) & \dots & g_j(z_{bn})\end{bmatrix}'$, then the function $\boldsymbol{g}_j(\boldsymbol{z}_b)$ with one predictor variable, symbolized as b, can be written in the following matrix form:

$$\boldsymbol{g}_j(\boldsymbol{z}_b)=\begin{bmatrix} g_j(z_{b1})\\ g_j(z_{b2})\\ \vdots \\ g_j(z_{bn})\end{bmatrix}=\begin{bmatrix} \alpha_{bj}z_{b1}+\frac{1}{2}\delta_{0bj}+\sum_{\theta=1}^{e}\delta_{\theta bj}\cos\theta z_{b1}\\ \alpha_{bj}z_{b2}+\frac{1}{2}\delta_{0bj}+\sum_{\theta=1}^{e}\delta_{\theta bj}\cos\theta z_{b2}\\ \vdots \\ \alpha_{bj}z_{bn}+\frac{1}{2}\delta_{0bj}+\sum_{\theta=1}^{e}\delta_{\theta bj}\cos\theta z_{bn}\end{bmatrix}$$

$$=\begin{bmatrix} \alpha_{bj}\mathrm{z}_{b1}+\frac{1}{2}\delta_{0bj}+\delta_{1bj}\cos 1z_{b1}+\delta_{2bj}\cos 2z_{b1}+\cdots+\delta_{ebj}\cos ez_{b1}\\ \alpha_{bj}\mathrm{z}_{b2}+\frac{1}{2}\delta_{0bj}+\delta_{1bj}\cos 1z_{b2}+\delta_{2bj}\cos 2z_{b2}+\cdots+\delta_{ebj}\cos ez_{b2}\\ \vdots \\ \alpha_{bj}\mathrm{z}_{bn}+\frac{1}{2}\delta_{0bj}+\delta_{1bj}\cos 1z_{bn}+\delta_{2bj}\cos 2z_{bn}+\cdots+\delta_{ebj}\cos ez_{bn}\end{bmatrix}$$

$$=\begin{bmatrix} z_{b1} & \frac{1}{2} & \cos 1z_{b1} & \cos 2z_{b1} & \dots & \cos ez_{b1}\\ z_{b2} & \frac{1}{2} & \cos 1z_{b2} & \cos 2z_{b2} & \dots & \cos ez_{b2}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ z_{bn} & \frac{1}{2} & \cos 1z_{bn} & \cos 2z_{bn} & \dots & \cos ez_{bn}\end{bmatrix}\begin{bmatrix}\alpha_{bj}\\ \delta_{0bj}\\ \delta_{1bj}\\ \delta_{2bj}\\ \vdots\\ \delta_{ebj}\end{bmatrix}$$

$$=\boldsymbol{Z}_{bj}\,\boldsymbol{\alpha}_{bj} \tag{16}$$

Likewise, the fourier series function for the q number of predictors nonparametric regression shown in Equation (17) for $b = 1,2,\dots,q$ predictors can be expressed as:

$$\begin{aligned}\boldsymbol{g}_j &= \sum_{b=1}^{q}\boldsymbol{g}_{bj}\\ &=\boldsymbol{g}_{1j}+\boldsymbol{g}_{2j}+\cdots+\boldsymbol{g}_{qj}\\ &=\boldsymbol{Z}_{1j}\,\boldsymbol{\alpha}_{1j}+\boldsymbol{Z}_{2j}\,\boldsymbol{\alpha}_{2j}+\cdots+\boldsymbol{Z}_{qj}\,\boldsymbol{\alpha}_{qj}\\ &=\sum_{b=1}^{q}\boldsymbol{Z}_{bj}\,\boldsymbol{\alpha}_{bj}\\ &=\boldsymbol{Z}_j\,\boldsymbol{\alpha}_j\end{aligned} \tag{17}$$

Where $\boldsymbol{\alpha}_j=[\alpha_{1j}\quad \delta_{01j}\quad \delta_{11j}\quad \delta_{21j}\quad \dots\quad \delta_{e1j}\quad \dots\quad \alpha_{qj}\quad \delta_{0qj}\quad \delta_{1qj}\quad \delta_{2qj}\quad \dots\quad \delta_{eqj}]'$ and $\boldsymbol{Z}_m =$

$$\begin{bmatrix} z_{11} & 1/2 & \cos 1z_{11} & \cos 2z_{11} & \dots & \cos ez_{11} & \dots & z_{q1} & 1/2 & \cos 1z_{11} & \cos 2z_{11} & \dots & \cos ez_{11}\\ z_{12} & 1/2 & \cos 1z_{12} & \cos 2z_{12} & \dots & \cos ez_{12} & \dots & z_{q2} & 1/2 & \cos 1z_{12} & \cos 2z_{12} & \dots & \cos ez_{12}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ z_{1n} & 1/2 & \cos 1z_{1n} & \cos 2z_{1n} & \dots & \cos ez_{1n} & \dots & z_{qn} & 1/2 & \cos 1z_{1n} & \cos 2z_{1n} & \dots & \cos ez_{1n}\end{bmatrix}$$

Based on Equation (17), the multiresponse Fourier series function is expressed in matrix form as shown in Equation (18).

$$\boldsymbol{g} = \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \vdots \\ \boldsymbol{g}_m \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{Z}_1\,\boldsymbol{\alpha}_1 \\ \boldsymbol{Z}_2\,\boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{Z}_m\,\boldsymbol{\alpha}_m \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{Z}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{Z}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{Z}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_m \end{bmatrix}$$

$$= \boldsymbol{Z}_{nm\times(2+e)qm}\boldsymbol{\alpha}_{(2+e)qm\times 1} \tag{18}$$

The function $h_j(t_{ci})$ is an estimate of the nonparametric regression function using the Gaussian kernel approach. The weight $V_{\psi_{cj}o}(t_{ci})$ represents the contribution of the o-th observation to the estimate at the point t_{ci}, calculated based on the proximity between t_{ci} and t_o, influenced by the bandwidth parameter ψ. The kernel function used is the Gaussian kernel, expressed in the form of Equation (19).

$$h_j(t_{ci}) = \sum_{o=1}^{n} n^{-1}\left[\frac{K_\psi(t-t_o)}{n^{-1}\sum_{j=1}^{n} K_\psi(t-t_j)}\right] y_{jo} = n^{-1}\sum_{o=1}^{n} V_{\psi_{cj}o}(t_{ci}) y_{jo} \tag{19}$$

If function $\boldsymbol{h}_j(\boldsymbol{t}_c) = \left[h_j(t_{c1}) \;\; h_j(t_{c2}) \;\; \dots \;\; h_j(t_{cn})\right]'$, then the function $\boldsymbol{h}_j(\boldsymbol{t}_c)$ with one predictor variable, symbolized as b, can be written in the following matrix form:

$$\boldsymbol{h}_j(\boldsymbol{t}_c) = \begin{bmatrix} h_j(t_{c1}) \\ h_j(t_{c2}) \\ \vdots \\ h_j(t_{cn}) \end{bmatrix} = \begin{bmatrix} n^{-1}\sum_{o=1}^{n} V_{\psi_{cj}o}(t_{c1}) y_{jo} \\ n^{-1}\sum_{o=1}^{n} V_{\psi_{cj}o}(t_{c2}) y_{jo} \\ \vdots \\ n^{-1}\sum_{o=1}^{n} V_{\psi_{cj}o}(t_{cn}) y_{jo} \end{bmatrix}$$

$$= \begin{bmatrix} n^{-1}V_{\psi_{cj}1}(t_{c1})y_{j1} + n^{-1}V_{\psi_{cj}2}(t_{c1})y_{j2} + \cdots + n^{-1}V_{\psi_{cj}n}(t_{c1})y_{jn} \\ n^{-1}V_{\psi_{cj}1}(t_{c2})y_{j1} + n^{-1}V_{\psi_{cj}2}(t_{c2})y_{j2} + \cdots + n^{-1}V_{\psi_{cj}n}(t_{c2})y_{jn} \\ \vdots \\ n^{-1}V_{\psi_{cj}1}(t_{cn})y_{j1} + n^{-1}V_{\psi_{cj}2}(t_{cn})y_{j2} + \cdots + n^{-1}V_{\psi_{cj}n}(t_{cn})y_{jn} \end{bmatrix}$$

$$= \begin{bmatrix} n^{-1}V_{\psi_{cj}1}(t_{c1}) & n^{-1}V_{\psi_{cj}2}(t_{c1}) & \dots & n^{-1}V_{\psi_{cj}n}(t_{c1}) \\ n^{-1}V_{\psi_{cj}1}(t_{c2}) & n^{-1}V_{\psi_{cj}2}(t_{c2}) & \dots & n^{-1}V_{\psi_{cj}n}(t_{c2}) \\ \vdots & \vdots & \ddots & \vdots \\ n^{-1}V_{\psi_{cj}1}(t_{cn}) & n^{-1}V_{\psi_{cj}2}(t_{cn}) & \dots & n^{-1}V_{\psi_{cj}n}(t_{c2}) \end{bmatrix} \begin{bmatrix} y_{j1} \\ y_{j2} \\ \vdots \\ y_{jn} \end{bmatrix}$$

$$= \boldsymbol{T}_{cj}\boldsymbol{y}_j \tag{20}$$

Next, based on the matrix structure derived in Equation (18) with a single predictor variable, the function h_j with r predictor variables, where $c = 1, 2, \dots r$, can be represented as shown in Equation (21).

$$\begin{aligned} \boldsymbol{h}_j &= \sum_{c=1}^{r} \boldsymbol{h}_{cj} \\ &= \boldsymbol{h}_{1j} + \boldsymbol{h}_{2j} + \cdots + \boldsymbol{h}_{rj} \\ &= \boldsymbol{T}_{1j}\boldsymbol{y}_j + \boldsymbol{T}_{2j}\boldsymbol{y}_j + \cdots + \boldsymbol{T}_{rj}\boldsymbol{y}_j \\ &= \left(\boldsymbol{T}_{1j} + \boldsymbol{T}_{2j} + \cdots + \boldsymbol{T}_{rj}\right)\boldsymbol{y}_j \\ &= \sum_{c=1}^{r} \boldsymbol{T}_{cj}\boldsymbol{y}_j \\ &= \boldsymbol{T}_j\boldsymbol{y}_j \end{aligned} \tag{21}$$

where $\boldsymbol{y}_j = [y_{11} \quad y_{12} \quad \cdots \quad y_{1n} \quad \cdots \quad y_{m1} \quad y_{m2} \quad \cdots \quad y_{mn}]'$ and

$$\boldsymbol{T}_j = \begin{bmatrix} n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}1}(t_{c1}) & n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}2}(t_{c1}) & \dots & n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}n}(t_{c1}) \\ n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}1}(t_{c2}) & n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}2}(t_{c2}) & \dots & n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}n}(t_{c2}) \\ \vdots & \vdots & \ddots & \vdots \\ n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}1}(t_{cn}) & n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}2}(t_{cn}) & \dots & n^{-1}\sum_{c=1}^{q} V_{\psi_{cj}n}(t_{cn}) \end{bmatrix}.$$

Based on Equation (21), the multiresponse Fourier series function is expressed in matrix form as shown in Equation (22).

$$\begin{aligned}\boldsymbol{h} &= \begin{bmatrix}\boldsymbol{h}_1\\ \boldsymbol{h}_2\\ \vdots\\ \boldsymbol{h}_m\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{T}_1\boldsymbol{y}_1\\ \boldsymbol{T}_2\boldsymbol{y}_2\\ \vdots\\ \boldsymbol{T}_j\boldsymbol{y}_j\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{T}_1 & 0 & \cdots & 0\\ 0 & \boldsymbol{T}_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{T}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{y}_1\\ \boldsymbol{y}_2\\ \vdots\\ \boldsymbol{y}_m\end{bmatrix}\\ &= \boldsymbol{T}_{nm\times nrm}\boldsymbol{y}_{nrm\times 1}\end{aligned} \tag{22}$$

3.2 Parameter Estimation with Weighted Least Square

The next step is to use the Weighted Least Square (WLS) method to estimate the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$. The weight matrix $\boldsymbol{W}$, a $nm \times nm$ variance-covariance matrix, is employed in this procedure. The particular structure of Matrix W is explained as follows:

$$\boldsymbol{W} = \begin{bmatrix}
\sigma_1^2 & 0 & \cdots & 0 & \rho\sigma_1\sigma_2 & 0 & \cdots & 0 & \cdots & \rho\sigma_1\sigma_m & 0 & \cdots & 0\\
0 & \sigma_1^2 & \cdots & 0 & 0 & \rho\sigma_1\sigma_2 & \cdots & 0 & \cdots & 0 & \rho\sigma_1\sigma_m & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \sigma_1^2 & 0 & 0 & \cdots & \rho\sigma_1\sigma_2 & \cdots & 0 & 0 & \cdots & \rho\sigma_1\sigma_m\\
\rho\sigma_2\sigma_1 & 0 & \cdots & 0 & \sigma_2^2 & 0 & \cdots & 0 & \cdots & \rho\sigma_1\sigma_m & 0 & \cdots & 0\\
0 & \rho\sigma_2\sigma_1 & \cdots & 0 & 0 & \sigma_2^2 & \cdots & 0 & \cdots & 0 & \rho\sigma_1\sigma_m & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \rho\sigma_2\sigma_1 & 0 & 0 & \cdots & \sigma_2^2 & \cdots & 0 & 0 & \cdots & \rho\sigma_1\sigma_m\\
 & \vdots & & & & \vdots & & & \ddots & & & & \\
\rho\sigma_m\sigma_1 & 0 & \cdots & 0 & \rho\sigma_m\sigma_2 & 0 & \cdots & 0 & \cdots & \sigma_m^2 & 0 & \cdots & 0\\
0 & \rho\sigma_m\sigma_1 & \cdots & 0 & 0 & \rho\sigma_m\sigma_2 & \cdots & 0 & \cdots & 0 & \sigma_m^2 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & \rho\sigma_m\sigma_1 & 0 & 0 & \cdots & \rho\sigma_m\sigma_2 & \cdots & 0 & 0 & \cdots & \sigma_m^2
\end{bmatrix}$$

Equation (22) can be shown in vector and matrix form as follows if Equation (14) provides the truncated linear spline regression curve $\boldsymbol{f}$, Equation (17) provides the Fourier series curve $\boldsymbol{g}$, and Equation (21) provides the kernel regression curve $\boldsymbol{h}$.

$$\begin{aligned}\boldsymbol{y} &= \boldsymbol{f} + \boldsymbol{g} + \boldsymbol{h} + \boldsymbol{\varepsilon}\\ \boldsymbol{y} &= \boldsymbol{U\beta} + \boldsymbol{Z\alpha} + \boldsymbol{Ty} + \boldsymbol{\varepsilon}\end{aligned} \tag{22}$$

where $\boldsymbol{y} = [\boldsymbol{y_1} \quad \boldsymbol{y_2} \quad \dots \quad \boldsymbol{y_m}]' = [y_{11}\, y_{12} \cdots y_{1n} \vdots y_{21}\, y_{22} \cdots y_{2n} \vdots y_{m1}\, y_{m2} \cdots y_{mn}]'$ and $\boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_1 \quad \boldsymbol{\varepsilon}_2 \quad \dots \quad \boldsymbol{\varepsilon}_m]' = [\varepsilon_{11}\, \varepsilon_{12} \cdots \varepsilon_{1n} \vdots \varepsilon_{21}\, \varepsilon_{22} \cdots \varepsilon_{2n} \vdots \varepsilon_{m1}\, \varepsilon_{m2} \cdots \varepsilon_{mn}]'$. Based on Equation (22), Equation (23) is obtained.

$$\begin{aligned}\boldsymbol{\epsilon} &= \boldsymbol{y} - \boldsymbol{f} - \boldsymbol{g} - \boldsymbol{h}\\ &= \boldsymbol{y} - \boldsymbol{U\beta} - \boldsymbol{Z\alpha} - \boldsymbol{Ty}\\ &= \boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha}\,;\ \boldsymbol{y}^* = (\boldsymbol{I} - \boldsymbol{T})\boldsymbol{y}\end{aligned} \tag{23}$$

Let $\boldsymbol{W}$ be a symmetric positive definite weighting matrix. The WLS estimator $(\widehat{\boldsymbol{\beta}},\widehat{\boldsymbol{\alpha}})$ is the solution to the following minimazation problem:

$$\min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\{\boldsymbol{\epsilon}'\boldsymbol{W}\boldsymbol{\epsilon}\} = \min_{\boldsymbol{\beta},\boldsymbol{\alpha}}\{(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})'\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})\}$$

Partial parameter derivation is used to optimize using the WLS method.

$$\begin{aligned}\boldsymbol{Q} &= \boldsymbol{\varepsilon}^T\boldsymbol{W}\boldsymbol{\varepsilon}\\ &= (\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})'\boldsymbol{W}(\boldsymbol{y}^* - \boldsymbol{U\beta} - \boldsymbol{Z\alpha})\\ &= \left(\boldsymbol{y}^{*'} - \boldsymbol{\beta}'\boldsymbol{U}' - \boldsymbol{\alpha}'\boldsymbol{Z}'\right)'(\boldsymbol{Wy}^* - \boldsymbol{WU\beta} - \boldsymbol{WZ\alpha})\\ &= \boldsymbol{y}^{*'}\boldsymbol{Wy}^* - \boldsymbol{y}^{*'}\boldsymbol{WU\beta} - \boldsymbol{y}^{*'}\boldsymbol{WZ\alpha} - \boldsymbol{\beta}'\boldsymbol{U}'\boldsymbol{Wy}^* + \boldsymbol{\beta}'\boldsymbol{U}'\boldsymbol{WU\beta} + \boldsymbol{\beta}'\boldsymbol{U}'\boldsymbol{WZ\alpha}\\ &\quad - \boldsymbol{\alpha}'\boldsymbol{Z}'\boldsymbol{Wy}^* + \boldsymbol{\alpha}'\boldsymbol{Z}'\boldsymbol{WU\beta} + \boldsymbol{\alpha}'\boldsymbol{Z}'\boldsymbol{WZ\alpha}\\ &= \boldsymbol{y}^{*'}\boldsymbol{Wy}^* - 2\boldsymbol{\beta}'\boldsymbol{U}'\boldsymbol{Wy}^* - 2\boldsymbol{\alpha}'\boldsymbol{Z}'\boldsymbol{Wy}^* + 2\boldsymbol{\beta}'\boldsymbol{U}'\boldsymbol{WZ\alpha} + \boldsymbol{\beta}'\boldsymbol{U}'\boldsymbol{WU\beta} + \boldsymbol{\alpha}'\boldsymbol{Z}'\boldsymbol{WZ\alpha}\end{aligned} \tag{24}$$

Then, the first derivative against $\boldsymbol{\beta}$ is set to zero as follows.

$$\begin{aligned}\frac{\partial(\boldsymbol{Q})}{\partial\boldsymbol{\beta}} &= 0\\ -2\boldsymbol{U}'\boldsymbol{Wy}^* + 2\boldsymbol{U}'\boldsymbol{WZ}\widehat{\boldsymbol{\alpha}} + 2\boldsymbol{U}'\boldsymbol{WU}\widehat{\boldsymbol{\beta}} &= \boldsymbol{0}\\ \boldsymbol{U}'\boldsymbol{WU}\widehat{\boldsymbol{\beta}} &= \boldsymbol{U}'\boldsymbol{Wy}^* - \boldsymbol{U}'\boldsymbol{WZ}\widehat{\boldsymbol{\alpha}}\\ \widehat{\boldsymbol{\beta}} &= (\boldsymbol{U}'\boldsymbol{WU})^{-1}(\boldsymbol{U}'\boldsymbol{Wy}^* - \boldsymbol{U}'\boldsymbol{WZ}\widehat{\boldsymbol{\alpha}})\end{aligned} \tag{25}$$

Then, the first derivative of Equation (24) againts $\boldsymbol{\alpha}$ is set to zero.

$$\begin{aligned}\frac{\partial(\boldsymbol{Q})}{\partial\boldsymbol{\alpha}} &= 0\\ -2\boldsymbol{Z}'\boldsymbol{Wy}^* + 2\boldsymbol{Z}'\boldsymbol{WU\beta} + 2\boldsymbol{Z}'\boldsymbol{WZ}\widehat{\boldsymbol{\alpha}} &= 0\\ \boldsymbol{Z}'\boldsymbol{WZ}\widehat{\boldsymbol{\alpha}} &= \boldsymbol{Z}'\boldsymbol{Wy}^* - \boldsymbol{Z}'\boldsymbol{WU}\widehat{\boldsymbol{\beta}}\\ \widehat{\boldsymbol{\alpha}} &= (\boldsymbol{Z}'\boldsymbol{WZ})^{-1}\left(\boldsymbol{Z}'\boldsymbol{Wy}^* - \boldsymbol{Z}'\boldsymbol{WU}\widehat{\boldsymbol{\beta}}\right)\end{aligned} \tag{26}$$

Equations (25) and (26) still contain parameters, so substitution needs to be performed. The first step, Equation (25) will be transformed into the following form:

$$\begin{aligned}\widehat{\boldsymbol{\beta}} &= (\boldsymbol{U}'\boldsymbol{WU})^{-1}(\boldsymbol{U}'\boldsymbol{Wy}^* - \boldsymbol{U}'\boldsymbol{WZ}\widehat{\boldsymbol{\alpha}})\\ \widehat{\boldsymbol{\beta}} &= \boldsymbol{Cy}^* - \boldsymbol{CZ}\widehat{\boldsymbol{\alpha}}\end{aligned} \tag{27}$$

where $\boldsymbol{C} = (\boldsymbol{U}'\boldsymbol{WU})^{-1}\boldsymbol{U}'\boldsymbol{W}$.

Meanwhile, Equation (26) will be transformed into Equation (28).

$$\begin{aligned}\widehat{\boldsymbol{\alpha}} &= (\boldsymbol{Z}'\boldsymbol{WZ})^{-1}\left(\boldsymbol{Z}'\boldsymbol{Wy}^* - \boldsymbol{Z}'\boldsymbol{WU}\widehat{\boldsymbol{\beta}}\right)\\ \widehat{\boldsymbol{\alpha}} &= \boldsymbol{Dy}^* - \boldsymbol{DU}\widehat{\boldsymbol{\beta}}\end{aligned} \tag{28}$$

where $\boldsymbol{D} = (\boldsymbol{Z}'\boldsymbol{WZ})^{-1}\boldsymbol{Z}'\boldsymbol{W}$.

Then, Equation (27) will be substituted into Equation (28):

$$\begin{aligned}\widehat{\boldsymbol{\alpha}} &= \boldsymbol{Dy}^* - \boldsymbol{DU}(\boldsymbol{Cy}^* - \boldsymbol{CZ}\widehat{\boldsymbol{\alpha}})\\ \widehat{\boldsymbol{\alpha}} &= \boldsymbol{Dy}^* - \boldsymbol{DUCy}^* + \boldsymbol{DUCZ}\widehat{\boldsymbol{\alpha}}\\ \widehat{\boldsymbol{\alpha}} - \boldsymbol{DUCZ}\widehat{\boldsymbol{\alpha}} &= \boldsymbol{Dy}^* - \boldsymbol{DUCy}^*\\ (\boldsymbol{I} - \boldsymbol{DUCZ})\widehat{\boldsymbol{\alpha}} &= (\boldsymbol{D} - \boldsymbol{DUC})\boldsymbol{y}^*\\ \widehat{\boldsymbol{\alpha}} &= (\boldsymbol{I} - \boldsymbol{DUCZ})^{-1}(\boldsymbol{D} - \boldsymbol{DUC})\boldsymbol{y}^*\\ \widehat{\boldsymbol{\alpha}} &= (\boldsymbol{I} - \boldsymbol{DUCZ})^{-1}(\boldsymbol{D} - \boldsymbol{DUC})(\boldsymbol{I} - \boldsymbol{T})\boldsymbol{y}\\ \widehat{\boldsymbol{\alpha}} &= \boldsymbol{Ay}\end{aligned} \tag{29}$$

with $\boldsymbol{A} = (\boldsymbol{I} - \boldsymbol{DUCZ})^{-1}(\boldsymbol{D} - \boldsymbol{DUC})(\boldsymbol{I} - \boldsymbol{T})$.

Then, Equation (29) can be substituted into Equation (27):

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \boldsymbol{C}\boldsymbol{y}^* - \boldsymbol{C}\boldsymbol{Z}\widehat{\boldsymbol{\alpha}} \\
\widehat{\boldsymbol{\beta}} &= \boldsymbol{C}\boldsymbol{y}^* - \boldsymbol{C}\boldsymbol{Z}((\boldsymbol{I}-\boldsymbol{DUCZ})^{-1}(\boldsymbol{D}-\boldsymbol{DUC})\boldsymbol{y}^*) \\
\widehat{\boldsymbol{\beta}} &= \left(\boldsymbol{C}-\boldsymbol{CZ}\left((\boldsymbol{I}-\boldsymbol{DUCZ})^{-1}(\boldsymbol{D}-\boldsymbol{DUC})\right)\right)\boldsymbol{y}^* \\
\widehat{\boldsymbol{\beta}} &= \left(\boldsymbol{C}-\boldsymbol{CZ}\left((\boldsymbol{I}-\boldsymbol{DUCZ})^{-1}(\boldsymbol{D}-\boldsymbol{DUC})\right)\right)(\boldsymbol{I}-\boldsymbol{T})\boldsymbol{y} \\
\widehat{\boldsymbol{\beta}} &= \boldsymbol{B}\boldsymbol{y}
\end{aligned} \tag{30}$$

with $\boldsymbol{B} = \left(\boldsymbol{C}-\boldsymbol{CZ}\left((\boldsymbol{I}-\boldsymbol{DUCZ})^{-1}(\boldsymbol{D}-\boldsymbol{DUC})\right)\right)(\boldsymbol{I}-\boldsymbol{T})$.

The mixed estimator of truncated spline, Fourier series, and kernel is derived as follows using the outcomes of Equations (29) and (30).

$$\begin{aligned}
\widehat{\boldsymbol{\mu}} &= \widehat{\boldsymbol{f}} + \widehat{\boldsymbol{g}} + \widehat{\boldsymbol{h}} \\
&= \boldsymbol{U}\widehat{\boldsymbol{\beta}} + \boldsymbol{Z}\widehat{\boldsymbol{\alpha}} + \boldsymbol{T}\boldsymbol{y} \\
&= \boldsymbol{UBy} + \boldsymbol{ZAy} + \boldsymbol{Ty} \\
&= \boldsymbol{Ey}
\end{aligned} \tag{31}$$

with $\boldsymbol{E} = \boldsymbol{UB} + \boldsymbol{ZA} + \boldsymbol{T}$.

3.3 Application of Mixed Estimator Model to Poverty Index Data

Before modeling using multi-response nonparametric regression analysis, the Pearson correlation value between response variables was first calculated, as shown in Table 2 below.

Table 2. Pearson Correlation Each Responses

	y_1	y_2	y_3
y_1	1	0.864	0.676
y_2	0.864	1	0.942
y_3	0.676	0.942	1

The correlation between response variables is very high. This finding indicates that the three response variables are significantly correlated with each other and this needs to be considered in the construction of a multi-response model. To observe the relationship pattern between the three response variables and each predictor variable, it can be done through scatterplot visualization. The scatterplot graph that describes the relationship between each response variable and each predictor variable is presented in Figure 1-3.

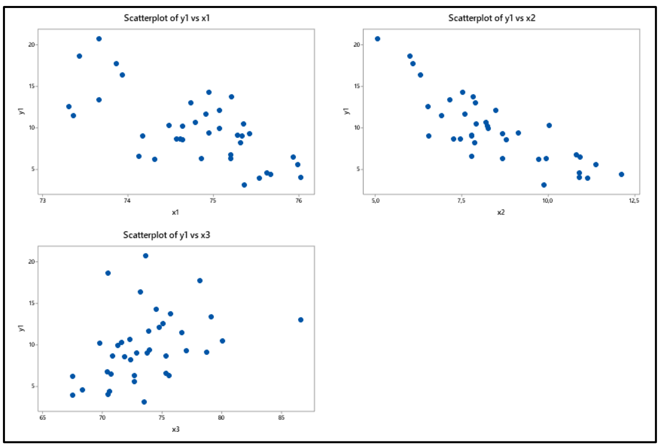

Figure 1. Scatterplot Percentage of Poor People (Y_1) with Each Predictors

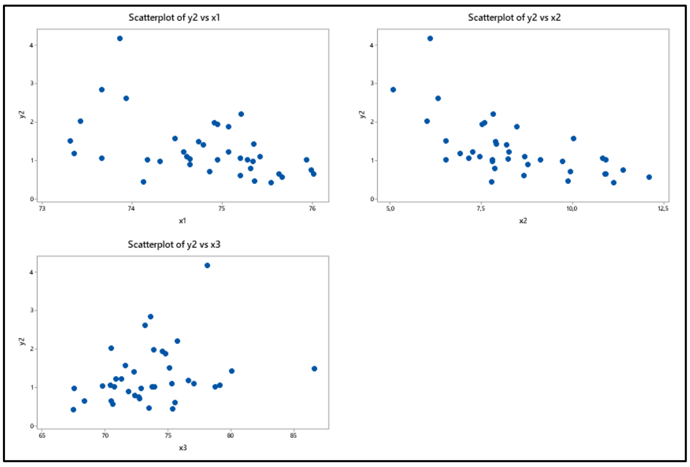

Figure 2. Scatterplot Poverty Depth Index (Y_2) with Each Predictors

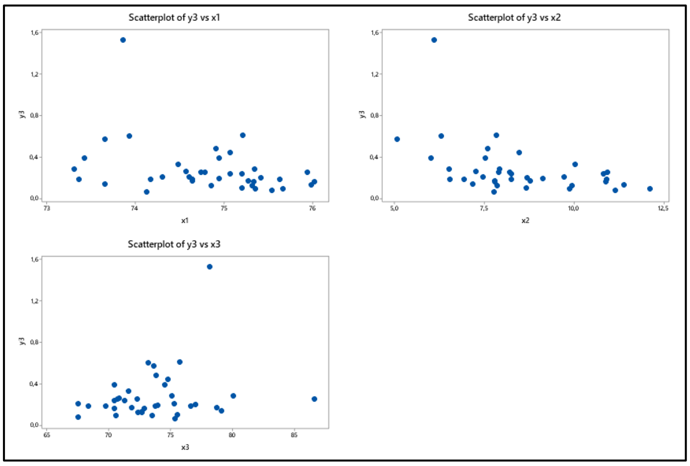

Figure 3. Scatterplot Poverty Severity Index (Y_3) with Each Predictors

The response variable Y_1 exhibits a clear negative relationship with predictors X_1 and X_2, particularly strong with X_2, while its association with X_3 lacks a discernible pattern. For Y_2, there is a general decreasing trend with X_1 and X_2, though less pronounced than in Y_1, and a slight upward trend with X_3, although non-linear. In contrast, Y_3 shows no consistent relationship with any of the predictors, with most values concentrated in a low and narrow range.

Table 3. Summary of GCV values every configuration model

Model	Variables			Number of Knots	Number of Oscillations	GCV
	Truncated Spline	Fourier Series	Kernel			
A.1	X_1	X_2	X_3	1	1	1.899
				2	2	1.814*
A.2	X_1	X_3	X_2	1	1	2.475
				2	2	2.632
A.3	X_2	X_1	X_3	1	1	3.538
				2	2	3.452
A.4	X_2	X_3	X_1	1	1	3.505
				2	2	3.590
A.5	X_3	X_2	X_1	1	1	2.369
				2	2	2.119
A.6	X_3	X_1	X_2	1	1	1.997
				2	2	1.888
B	X_1, X_2, X_3			1		140.310
				2		148.764
C		X_1, X_2, X_3			1	699.812
					2	786.152
D			X_1, X_2, X_3			160.181

*The Smallest Value GCV

The table presents the Generalized Cross Validation (GCV) results for multiple nonparametric regression models involving various configurations of truncated spline, Fourier series, and kernel estimators. Models A.1 through A.6 represents hybrid structures, each combining one variable for truncated spline, one for Fourier series, and one for kernel smoothing, with evaluations conducted under two settings of model complexity: 1 knot with 1 oscillation, and 2 knots with 2 oscillations. Among these, Model A.1 demonstrates the best performance with a GCV of 1.814, indicating that the configuration with X_1 as truncated spline, X_2 as Fourier series, and X_3 as kernel yields the most accurate and parsimonious fit. This model yields a coefficient of determination (R^2) of 94.066%. Due to the satisfactory result of R^2, it can be said that the proposed model is able to describe the variance of the response variable through the predictor variables exceptionally well. The mixed estimator's superiority stems from its adaptability: splines capture local trends in X_1 (life expectancy), Fourier series model periodicity in X_2

(education), and kernel smooth noisy X_3 (labor data). This synergy reduces overfitting (variance) while maintaining sensitivity to localized patterns (bias). Figure 4 further validates the model's accuracy by comparing actual and predicted values for each response variable, demonstrating a close alignment with minimal residuals.

In contrast, Models B, C, and D represent homogeneous basis structures: Model B applies truncated spline to all predictors, Model C uses only Fourier series, and Model D uses only kernel estimators. These models produce significantly higher GCV values, with Model C showing the poorest performance (GCV = 699.812 and 786.152), followed by Model D (GCV = 160.181), and Model B (GCV = 140.310 and 148.764). The results indicate that fully homogeneous basis configurations are substantially less effective than mixed-basis models in capturing the structure of the data, reaffirming the advantage of selectively assigning basis functions based on variable characteristics.

Based on the parameter estimation results for each response variable for the 2024 Poverty Index data of East Java, the multiresponse nonparametric regressionmodel with a combined truncated spline, Fourier series, and kernel estimator can be written as.

$$\hat{y}_{1i} = 10.037 + 0.095u_i - 24.221(u_i - 6.422)_+ + 24.53(u_i - 6.457)_+ + 0.177z_i + \frac{1}{2}3.236 \times 10^{-11} + 0.137\cos z_i - 0.047\cos 2z_i + n^{-1}\sum_{o=1}^{38} V_{\psi_1 o}(t_i)y_{1o} + \epsilon_{1i} \quad (31)$$

Where $V_{\psi_1 o} = \left[\frac{K(\frac{t-t_o}{0.503})}{\sum_{j=1}^{38} K(\frac{t-t_j}{0.503})}\right]$

$$\hat{y}_{2i} = 0.075 + 0.028u_i - 3.978(u_i - 6.457)_+ + 4.003(u_i - 6.493)_+ + 0.035z_i + \frac{1}{2}5.921 \times 10^{-13} + 0.021\cos z_i + 0.025\cos 2z_i + n^{-1}\sum_{o=1}^{38} V_{\psi_2 o}(t_i)y_{2o} + \epsilon_{2i} \quad (32)$$

Where $V_{\psi_2 o} = \left[\frac{K(\frac{t-t_o}{0.653})}{\sum_{j=1}^{38} K(\frac{t-t_j}{0.653})}\right]$

$$\hat{y}_{3i} = 0.012 + 0.002u_i - 0.343(u_i - 6.493)_+ + 0.351(u_i - 6.528)_+ + 0.146z_i + \frac{1}{2}1.308 \times 10^{-12} + 0.012\cos z_i + 0.031\cos 2z_i + n^{-1}\sum_{o=1}^{38} V_{\psi_2 o}(t_i)y_{3o} + \epsilon_{3i} \quad (33)$$

Where $V_{\psi_3 o} = \left[\frac{K(\frac{t-t_o}{0.64})}{\sum_{j=1}^{38} K(\frac{t-t_j}{0.64})}\right]$

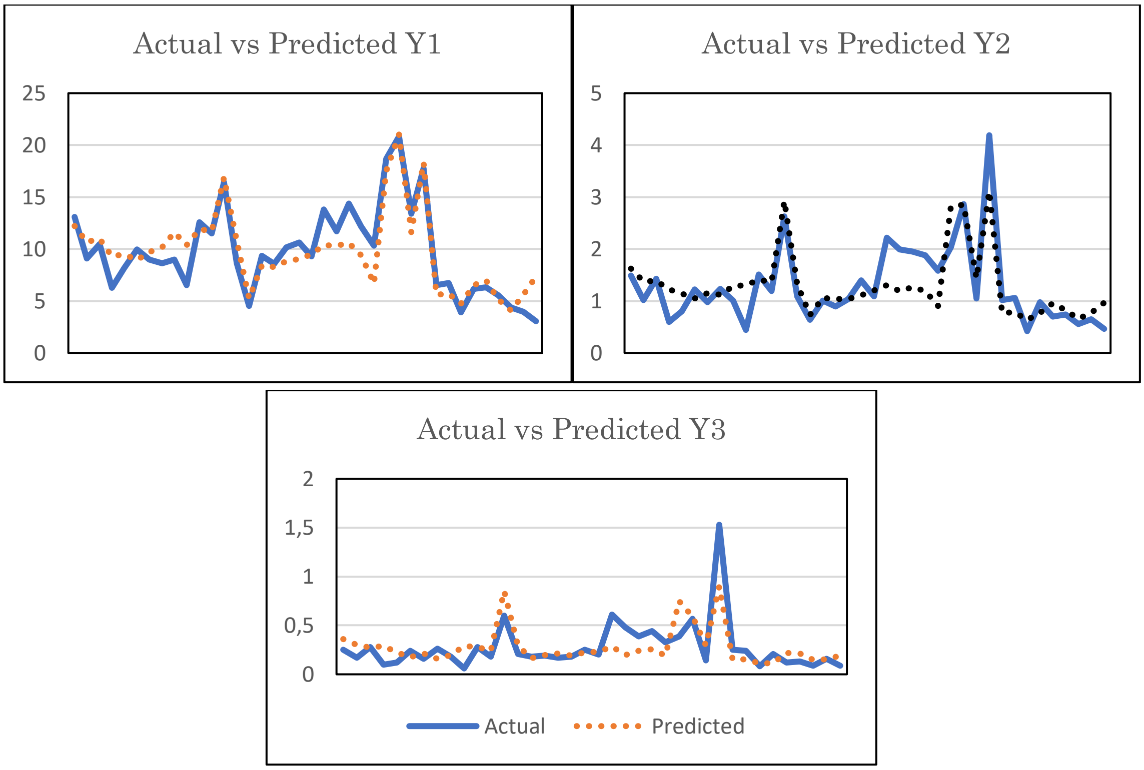

Figure 4. Actual vs Predicted Every Response

The results confirm that the mixed estimator model provides more accurate and reliable estimates than homogeneous basis models. This advantage stems from its ability to assign appropriate estimation methods based on each predictor's data pattern. However, the proposed model is not without limitations. First, it is computationally more demanding, especially when optimizing over multiple parameter spaces (e.g., knot numbers, harmonic orders, bandwidths). Second, the efficiency of the parameter tuning approach is crucial to the model's performance; mistakes in knot or bandwidth selection could weaken the estimator's reliability.

Moreover, this method may be sensitive to multicollinearity among predictor variables if not addressed carefully in the weighting matrix. Despite these challenges, the mixed estimator provides a flexible and adaptable modeling framework. It is particularly suitable for practitioners analyzing complex multivariate systems where traditional univariate or homogeneous regression approaches fall short.

4. CONCLUSIONS

Based on the previously conducted analysis, Weighted Least Sqaure (WLS) optimization is used to obtain the multiresponse nonparametric regression model with a mixed truncated spline, fourier series and kernel estimator.

$$\begin{aligned}\hat{\boldsymbol{\mu}} &= \hat{\boldsymbol{f}} + \hat{\boldsymbol{g}} + \hat{\boldsymbol{h}} \\ &= \boldsymbol{U}\hat{\boldsymbol{\beta}} + \boldsymbol{Z}\hat{\boldsymbol{\alpha}} + \boldsymbol{T}\boldsymbol{y} \\ &= \boldsymbol{U}\boldsymbol{B}\boldsymbol{y} + \boldsymbol{Z}\boldsymbol{A}\boldsymbol{y} + \boldsymbol{T}\boldsymbol{y} \\ &= \boldsymbol{E}\boldsymbol{y}\end{aligned}$$

with $\boldsymbol{E} = \boldsymbol{U}\boldsymbol{B} + \boldsymbol{Z}\boldsymbol{A} + \boldsymbol{T}$.

The proposed mixed estimator provides a robust framework for multiresponse nonparametric regression, combining the strengths of truncated spline, Fourier series, and kernel methods. Its adaptability to diverse data patterns makes it particularly valuable for applications in economics, public health, and environmental science, where multivariate outcomes are common. However, the model's computational complexity and sensitivity to parameter tuning remain challenges. Future research could explore adaptive weighting mechanisms and extensions to dynamic or high-dimensional data

5. RECOMMENDATIONS

Based on the findings of this study, several recommendations can be proposed to enhance the applicability and robustness of the mixed nonparametric regression model. First, incorporating an adaptive weighting mechanism to determine the relative contribution of truncated spline, Fourier series, and kernel components based on the underlying data characteristics could improve model flexibility and reduce the risk of overfitting. Second, the model's performance heavily depends on the selection of tuning parameters such as knot placement, harmonic order, and kernel bandwidth; thus, implementing cross-validation techniques such as generalized cross-validation is strongly recommended for optimal parameter selection. Third, given the static nature of the current formulation, extending the model to accommodate spatiotemporal or longitudinal data structures would increase its relevance for real-world applications involving dynamic multiresponse phenomena. These enhancements would further strengthen the model's capacity to capture complex response-predictor relationships across various domains.

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